

OPERATIONS RESEARCH

UNIT 1 Lecture 1



Presented By:

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Lecture Details:

**Introduction to Operation Research – Development,
Definition, Phases, Types of Models and Applications**

Operations Research/ME, 5 Semester.

LEARNING OUTCOMES

Upon completion of this lecture, student should be able to

- Explain characteristics of operations research
- Describe phases of operations research study
- List types of models of operations research study

Development of Operations Research

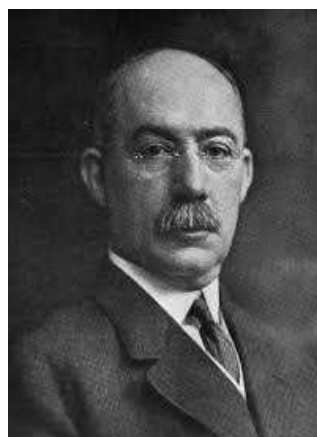
Pre World War II



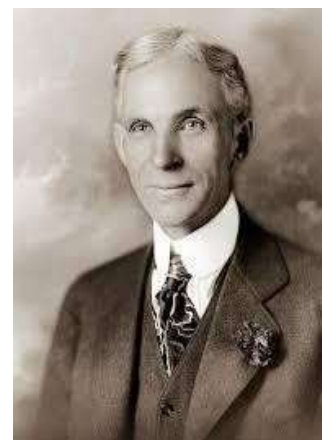
Industrial
Revolution



F W Taylor



Henry Gantt



F W Harris



A K Erlang

Division of Management
Functions (1820 - 1840)

Work Study
(1885)

Job Shop Scheduling
(1912 - 1915)

Inventory Control
(1915)

Queuing Theory
(1917)

Development of Operations Research

World War II



The term operations research was coined by McClosky and Trefthen in 1940 in the UK

Game Theory
(1944)



John Van Neumann

Development of Operations Research

Post World War II

OR Unit - Regional Research laboratory - 1949

Operations Research society of America - 1952

Institute of Management Sciences - 1953

OR Unit in Indian Statistical Institute - 1953

OR Society of India - 1957

International Federation of OR Societies - 1959

Definition of Operations Research

Operations research is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control. - Morse and Kimball

Operations research in the most general sense, can be characterized as the application of scientific methods, tools and techniques to problems involving the operations of systems so as to provide those in control of the operations with optimum solutions to the problems. - Churchman, Ackoff, Arnoff

Definition of Operations Research

Operations research is applied decision theory. It uses any scientific, mathematical or logical means to attempt to cope with the problems that confronts the executive when he tries to achieve a thoroughgoing rationality in dealing with his decision problems.

- Miller, Starr

Operations research is a scientific approach to problem solving for executive management

- H M Wagner

Operations research is the art of winning wars without actually fighting them.

- Auther Clark

Definition of Operations Research

Operations research is an experimental and applied science devoted to observing, understanding and predicting the behaviour of purposeful man-machine systems; and operations research works are actively engaged in applying this knowledge to practical problems in business, government and society.

- Operations Research Society of America

Operations research utilises the planned approach and an interdisciplinary team in order to represent complex functional relationships as mathematical models for the purpose of providing a quantitative basis for decision making and uncovering new problems for quantitative analysis

- Thierauf, Klekamp

Characteristics of Operations Research

The essential characteristics of operations research are

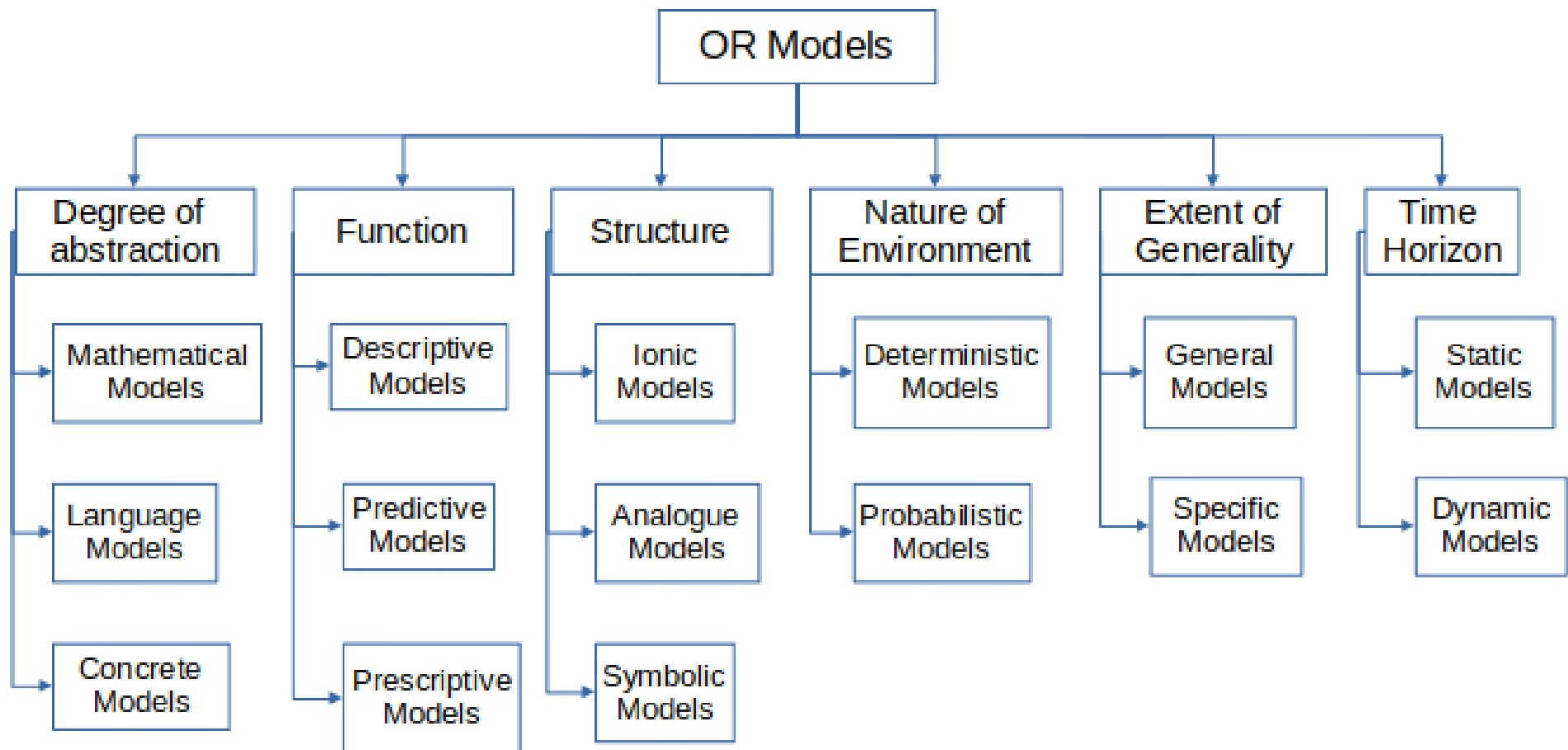
- Systems orientation
- Interdisciplinary teams
- Scientific method
- Uncovering new problems
- Improving quality of decisions
- Use of computer
- Quantitative solutions
- Human factors

Phases of Operations Research Study

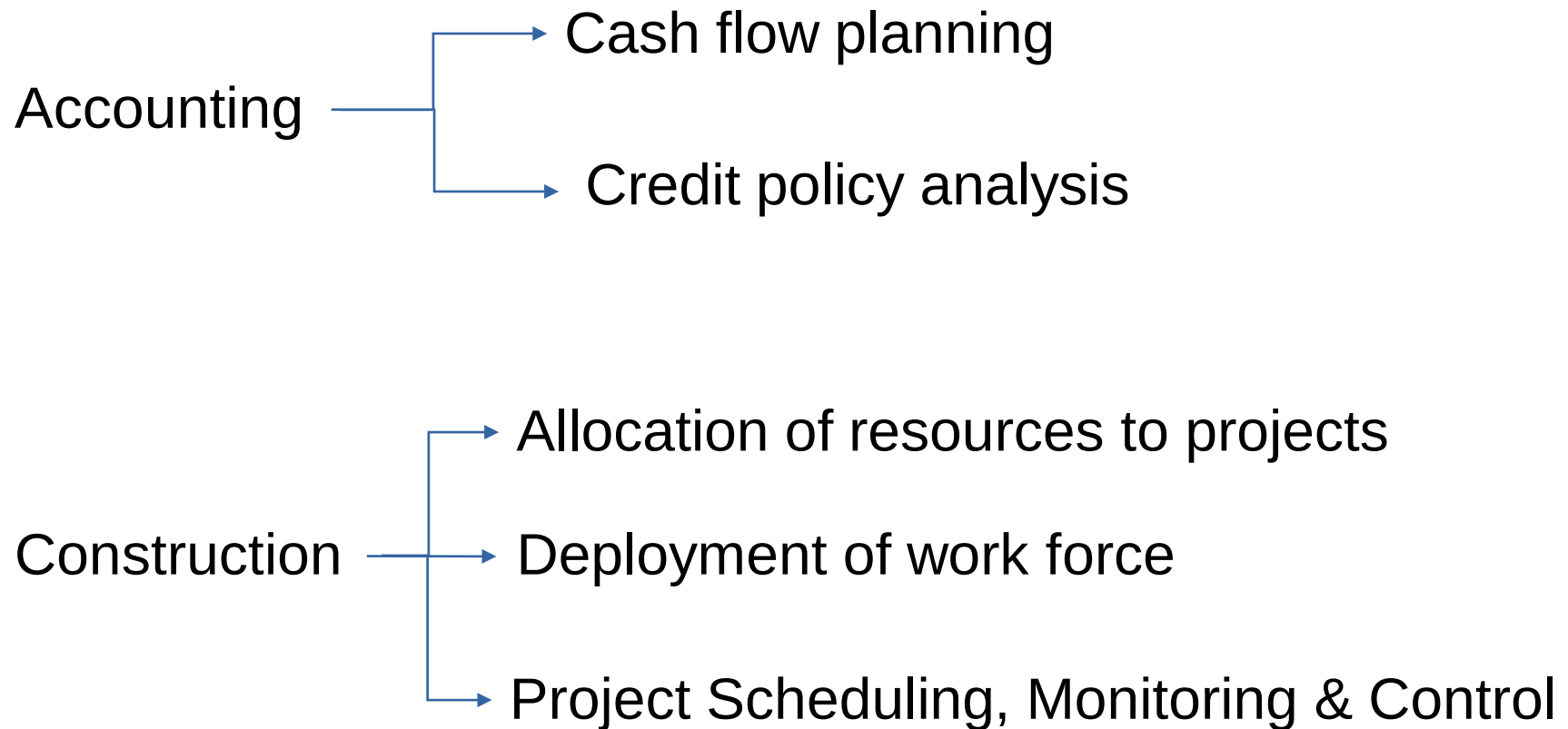
Principal phases for implementing OR in practice

- Definition of the problem
- Construction of the model
- Solution of the model
- Validation of the model
- Implementation of the solution

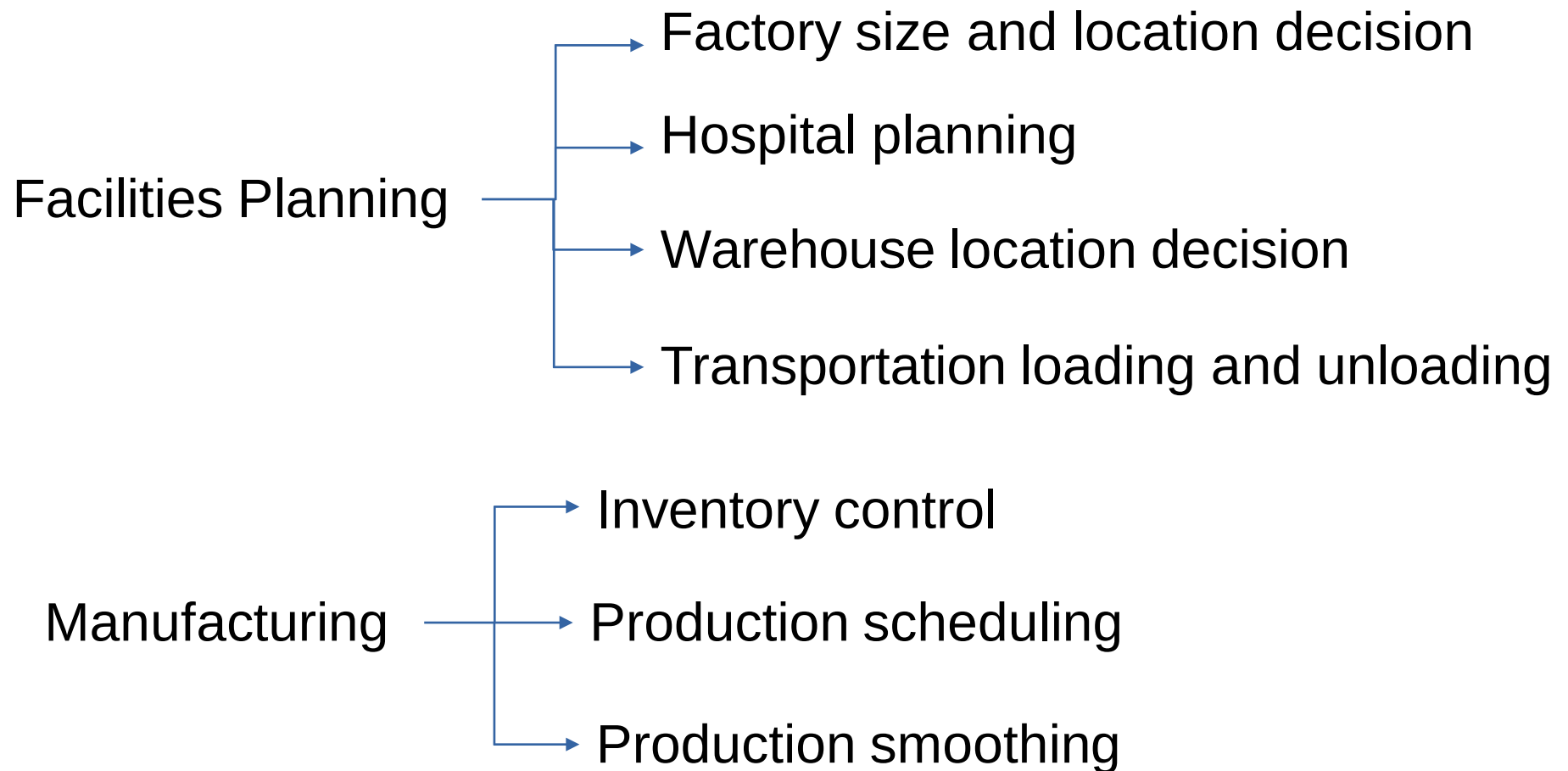
Operations Research Models



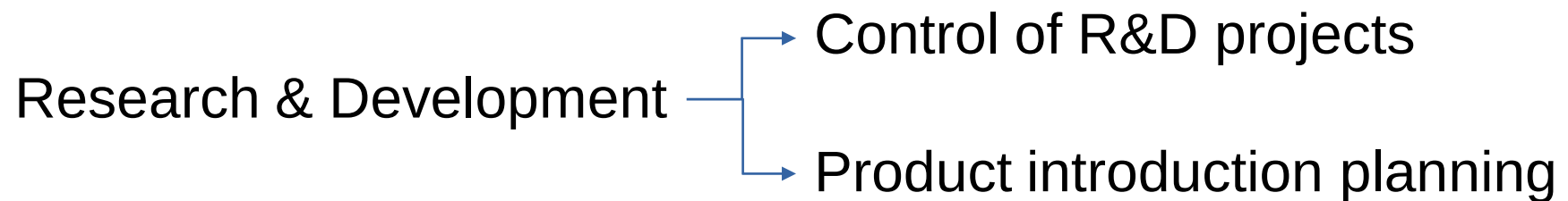
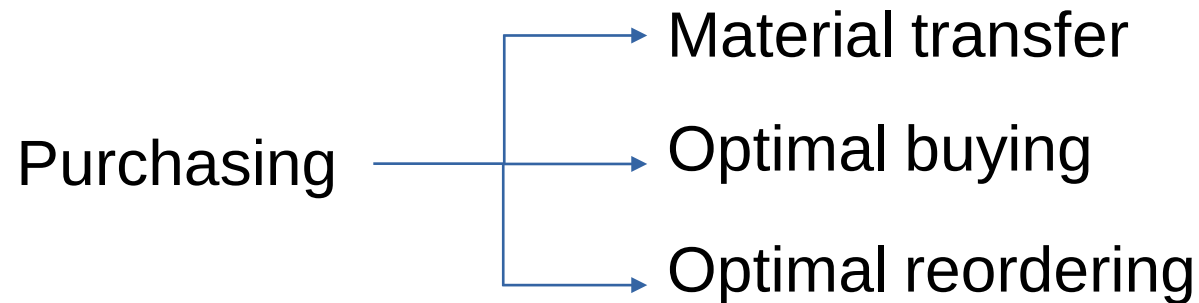
Applications of Operations Research



Applications of Operations Research



Applications of Operations Research



OPERATIONS RESEARCH

UNIT 1

Lecture 2



Presented By:

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Lecture Details:

Introduction to Linear Programming – Formulation

Solution of Linear Programming Problem – Graphical Method

Operations Research/ME, 5 Semester.

LEARNING OUTCOMES

At the end of this lecture, student should be able to

- Formulate an allocation problem as linear programming problem with suitable assumptions
- Solve linear programming problem using graphical method

Linear Programming (LP) - Introduction

- Linear programming is the tool, that can solve the problem of allocation of limited resources among the competitive activities in an optimal way.
- It is conceived by George Bernard Dantzig.
- The term linear indicates that all the mathematical functions involved are linear functions, and programming indicates the planning.
- This tool has a wide range of applications - allocation of production facilities in manufacturing scenario, allocation of natural resources to domestic needs, selection of shipping patterns, and so on.

Formulation of Linear Programming Problem

A manufacturer produces two variants of product A and B from two raw materials, M1 and M2. 1000 units of product A requires 6000 units of M1 and 1000 units of M2, whereas 1000 units of product B requires 4000 units of M1 and 2000 units of M2. Maximum availability of the raw materials M1 and M2 on a daily basis is 24000 and 6000 units respectively. Profit earned by the manufacturer per unit of product A is 5 rupees and per unit of product B is 4 rupees. Survey conducted by a third party indicates that the daily demand for product B cannot exceed that for product A by more than 1000 units. Also the maximum daily demand for product B is 2000 units. What should be the product mix that should be manufactured to maximize the daily profit.

Formulation of Linear Programming Problem

Let us assume that

X_1 = Number of units of product A made daily

X_2 = Number of units of product B made daily

Profit earned by the manufacturer

$$Z = 5X_1 + 4X_2$$

Based on the availability of raw material

$$6X_1 + 4X_2 \leq 24000 \quad \mathbf{M1}$$

$$X_1 + 2X_2 \leq 6000 \quad \mathbf{M2}$$

Formulation of Linear Programming Problem

From the demand restrictions

$$X_2 - X_1 \leq 1000$$

$$X_2 \leq 2000$$

Since the number of items produced cannot be negative

$$X_1 \geq 0$$

$$X_2 \geq 0$$

Formulation of Linear Programming Problem

Thus the linear programming model for the given problem is

$$\text{Maximize } Z = 5X_1 + 4X_2 \quad \text{Objective function}$$

Subject to

$$6X_1 + 4X_2 \leq 24000$$

$$X_1 + 2X_2 \leq 6000$$

$$-X_1 + X_2 \leq 1000$$

$$X_2 \leq 2000$$

Functional
constraints

And

$$X_1, X_2 \geq 0 \quad \text{Non negativity constraints}$$

X_1 and X_2 are called as **decision variables**

Properties of the Linear Programming Model

Linear programming model must satisfy the following basic properties

Linearity

The objective function and the constraints are linear.

Proportionality

The contribution of each decision variable in both the objective function and the constraints is directly proportional to the value of the variable.

Properties of the Linear Programming Model

Linear programming model must satisfy the following basic properties

Additivity

The total contribution of all the variables in the objective function and in the constraints is the direct sum of the individual contributions of each variable.

Certainty

All the objective and constraint coefficients of the linear programming model are deterministic or known constants.

Terminology for Solutions of the Model

Solution

Any specification of the values for the decision variables.

Feasible solution

Solution which satisfies all the constraints.

Infeasible solution

Solution which violates at least one constraint.

Feasible region

Collection of all feasible solutions.

Terminology for Solutions of the Model

Optimal solution

Feasible solution that has the most favorable value of the objective function.

Corner point feasible solution

Solution that lies at a corner of the feasible region.

Graphical Method - Linear Programming Solution

- Graphical method can be used to solve linear programming problems involving two decision variables.
- The method can be extended (with difficulty) to linear programming problems with three decision variables, but not more than three.
- For linear programming problems involving large number of decision variables, simplex method is used.
- Graphical procedure includes two steps:
 1. Determination of the feasible solution space
 2. Determination of the optimum solution from among all the feasible points in the solution space

Graphical Method – Example 1

Solve the following linear programming problem using graphical method

$$\text{Maximize } Z = 3X_1 + 5X_2$$

Subject to

$$X_1 \leq 4$$

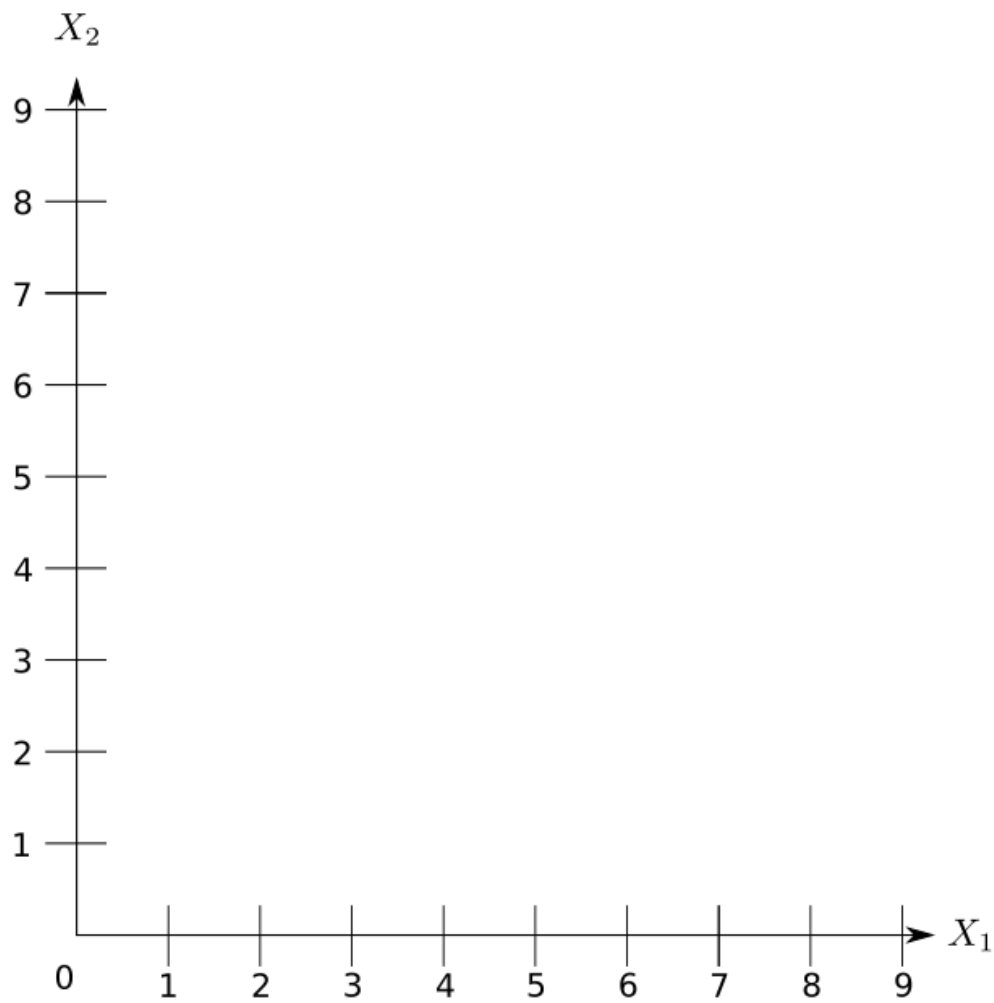
$$X_2 \leq 6$$

$$3X_1 + 2X_2 \leq 18$$

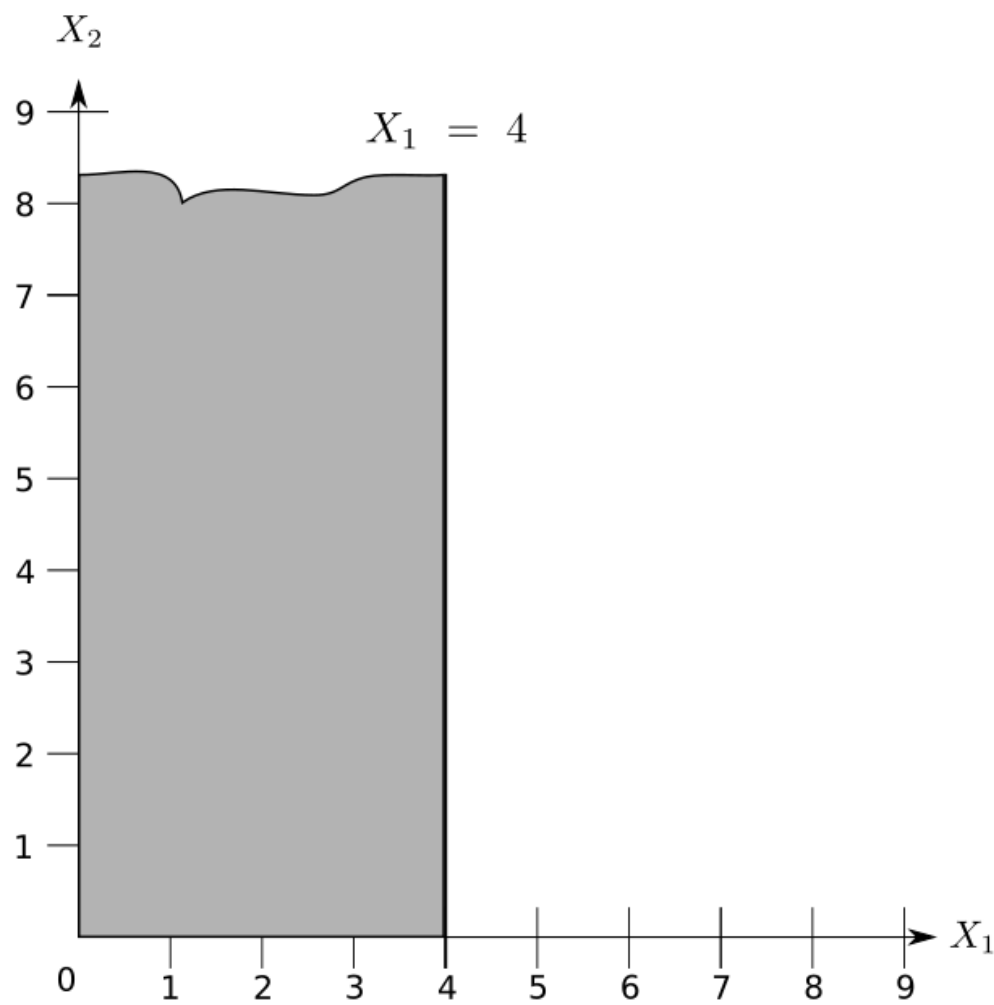
And

$$X_1, X_2 \geq 0$$

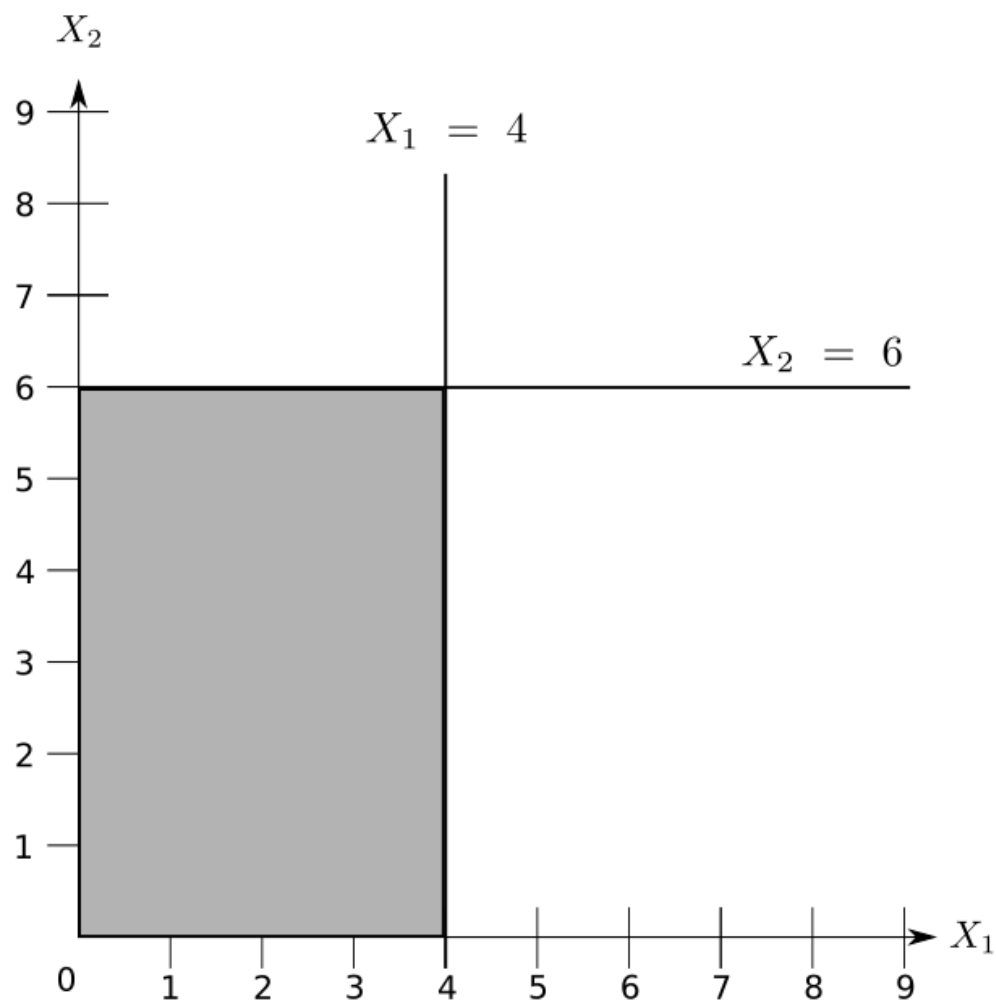
Graphical Method – Example



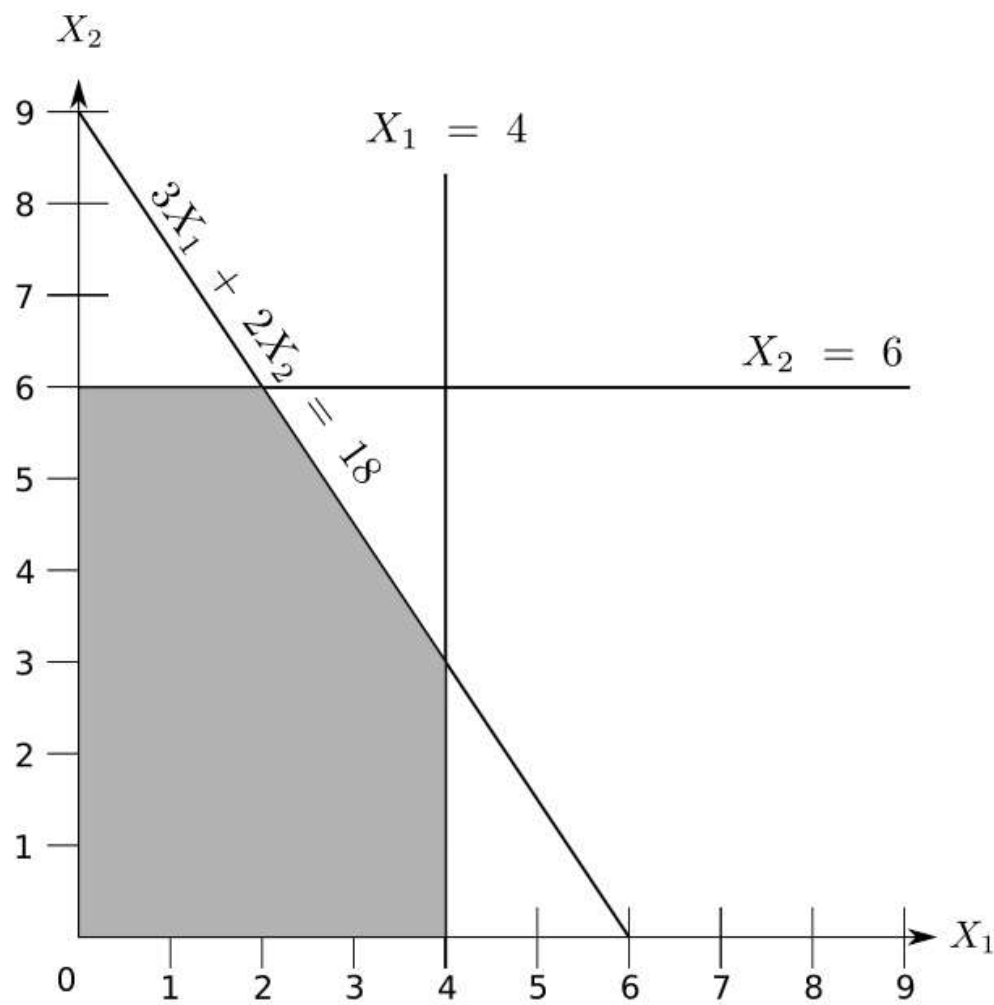
Graphical Method – Example



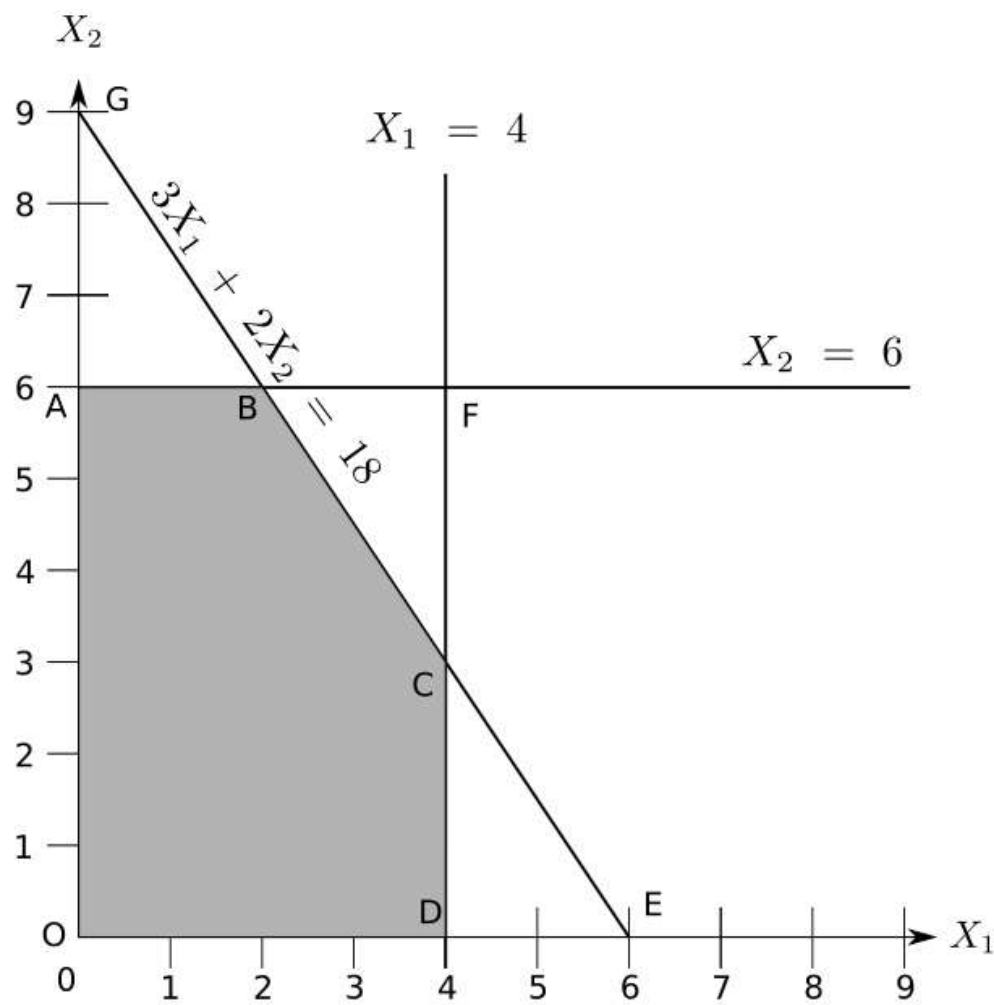
Graphical Method – Example



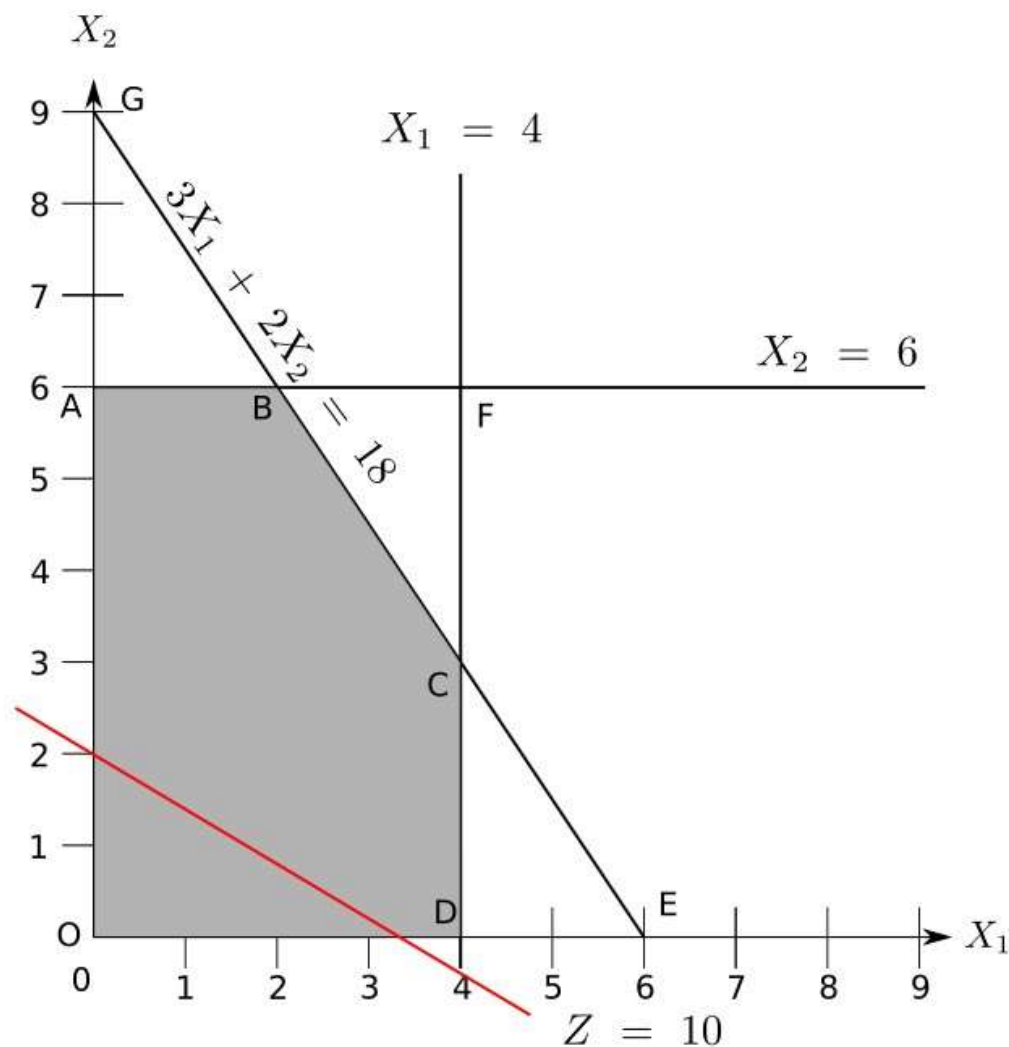
Graphical Method – Example



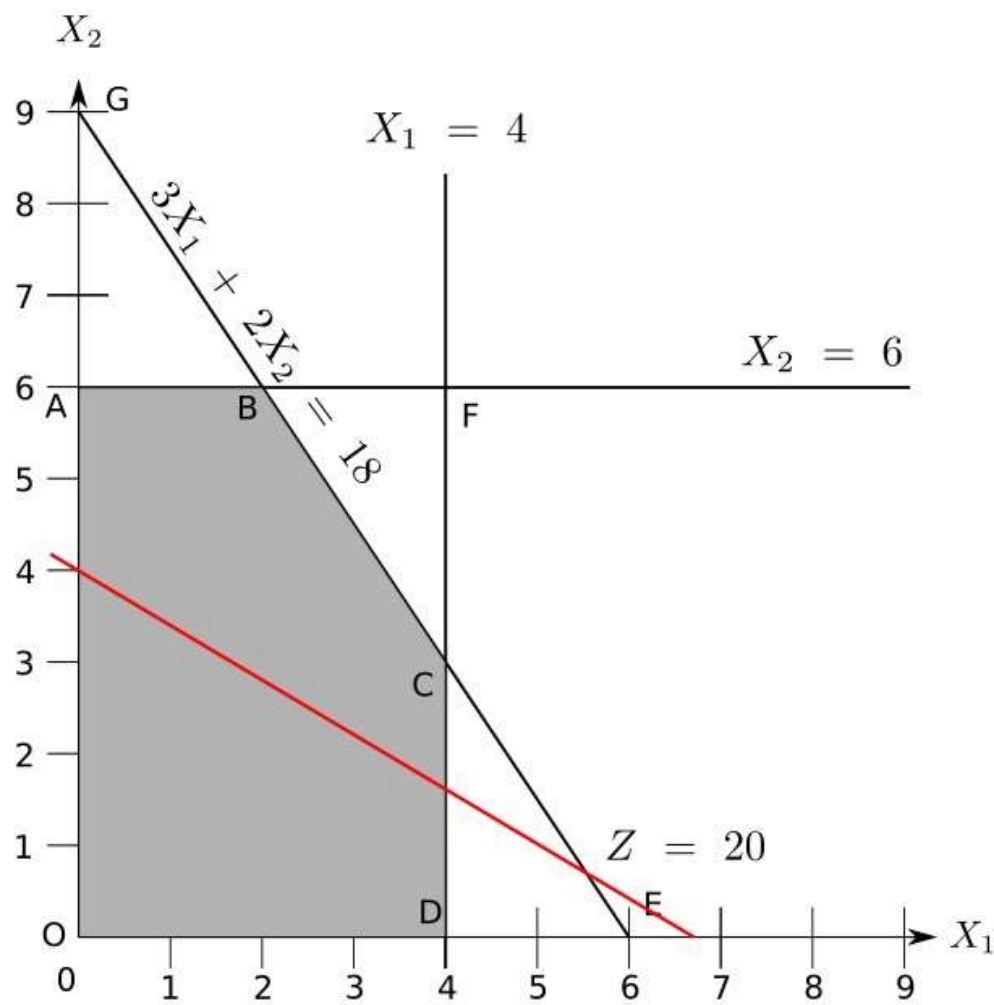
Graphical Method – Example



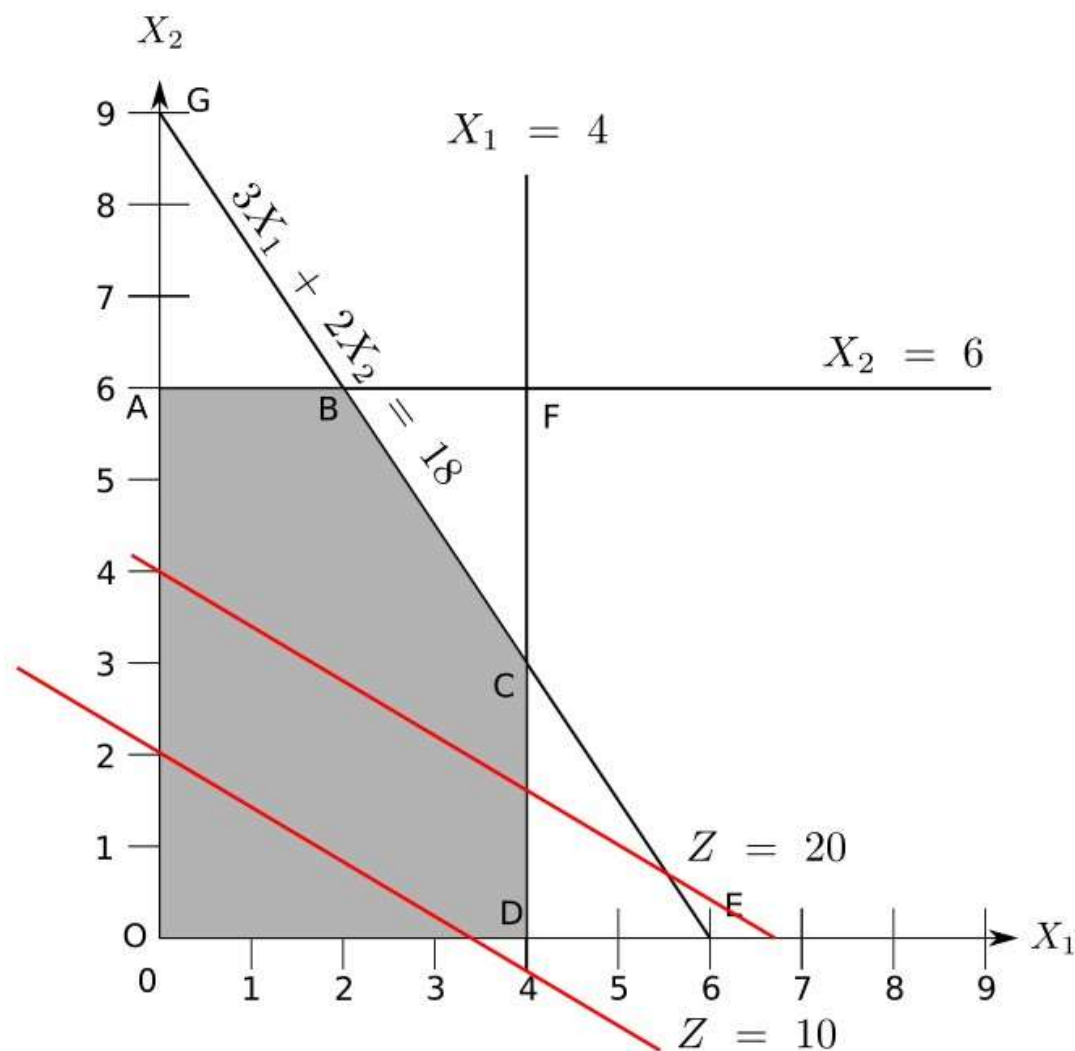
Graphical Method – Example



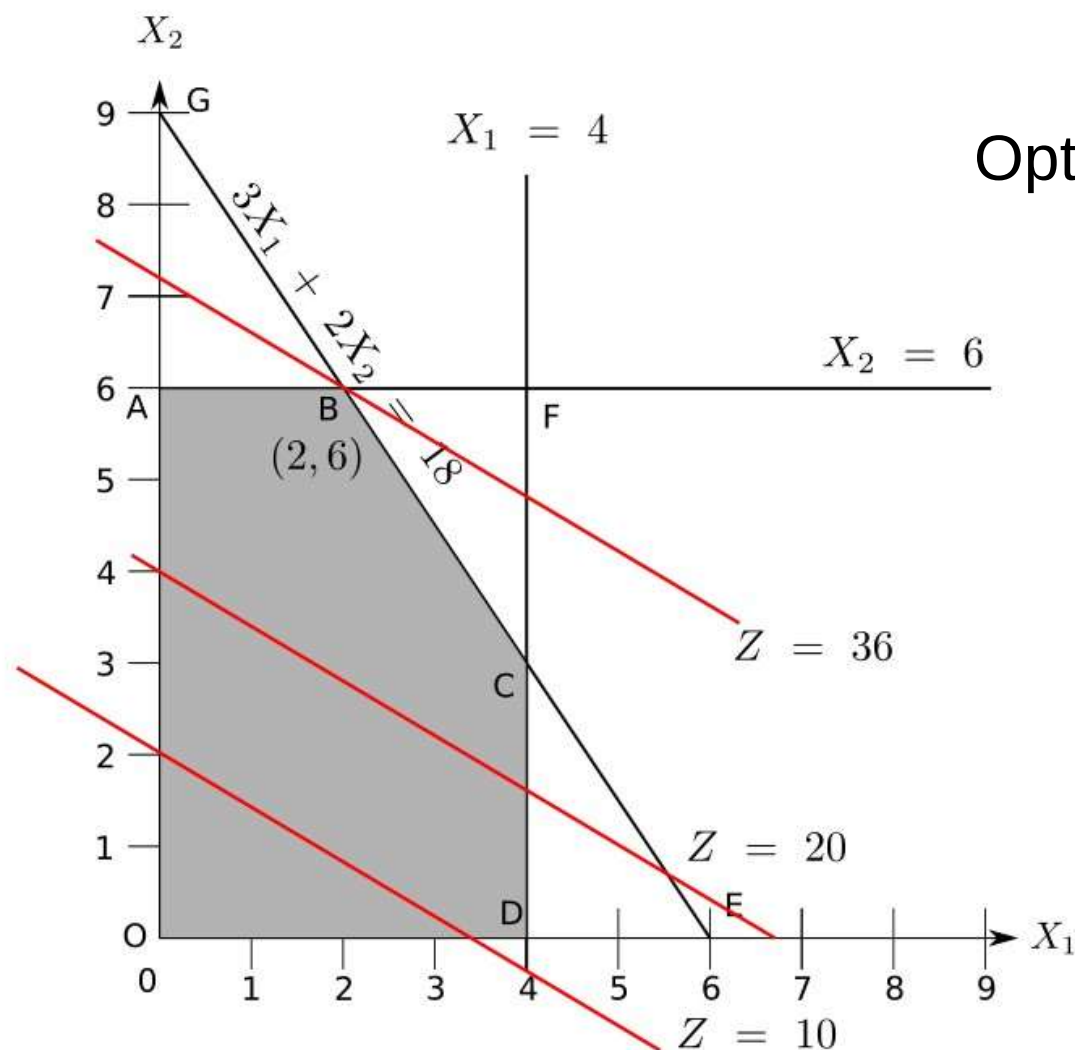
Graphical Method – Example



Graphical Method – Example



Graphical Method – Example



Optimal Solution

$$\begin{aligned} X_1 &= 2 \\ X_2 &= 6 \text{ with} \\ Z &= 36 \end{aligned}$$

Key Points – LP Problem

- Optimum LP solution is always associated with a corner point solution in such a way that the best corner point solution must be the optimal solution.
- Following statements can be derived from the above
 1. If problem has exactly one optimal solution, it must be a corner point solution.
 2. If the problem has multiple optimal solutions , at least two must be corner point solutions.

OPERATIONS RESEARCH

UNIT 1

Lecture 3



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Lecture Details:

Solution of Linear Programming Problem – Simplex Method

Operations Research/ME, 5 Semester.

LEARNING OUTCOMES

Upon completion of this lecture, student should be able to

- Describe the procedure for solving a linear programming problem using simplex algorithm
- Solve linear programming problems using simplex algorithm

Linear Programming (LP) – Simplex Method

- There are two requirements imposed on constraints of the linear programming problem, which are needed to develop the simplex method computations.
 - 1) All the constraints (except non negativity constraint) are equations with non negative right hand side.
 - 2) All the variables are non negative.

Simplex Method – Linear Programming Problem

Solve the given linear programming problem using simplex method

$$\text{Maximize } Z = 100X_1 + 60X_2 + 40X_3$$

Subject to

$$X_1 + X_2 + X_3 \leq 100$$

$$10X_1 + 4X_2 + 5X_3 \leq 600$$

$$2X_1 + 2X_2 + 6X_3 \leq 300$$

$$X_1, X_2, X_3 \geq 0$$

Linear Programming Problem – Simplex Method

Simplex Method
Step 1
Step 2
Step 3
Step 4
Step 5
Step 6
Step 7
Step 8
Step 9
Step 10
Step 11

Step 1 – Formulate the given problem into linear programming problem model

Step 2 – Ensure that the objective function is in maximization form. If not multiply it with -1.

Example:

$$\text{Minimize } Z = 2X_1 + 3X_2$$

$$\text{Maximize } -Z = -2X_1 - 3X_2$$

Linear Programming Problem – Simplex Method

Simplex Method
Step 1
Step 2
Step 3
Step 4
Step 5
Step 6
Step 7
Step 8
Step 9
Step 10
Step 11

Step 3 – Ensure that all the constraints have non negative right hand side. If not, multiply both the sides of constraint with -1.

Example:

$$5X_1 + 3X_2 \leq -10$$

$$-5X_1 - 3X_2 \geq 10$$

Linear Programming Problem – Simplex Method

Simplex Method
Step 1
Step 2
Step 3
Step 4
Step 5
Step 6
Step 7
Step 8
Step 9
Step 10
Step 11

Step 4 – Convert all the inequality constraints into equations by using slack, surplus and artificial variables as required on the left hand side of the inequality constraint.

For the given problem

$$X_1 + X_2 + X_3 + S_1 = 100$$

$$10X_1 + 4X_2 + 5X_3 + S_2 = 600$$

$$2X_1 + 2X_2 + 6X_3 + S_3 = 300$$

$$X_1, X_2, X_3, S_1, S_2, S_3 \geq 0$$

Linear Programming Problem – Simplex Method

Simplex Method
Step 1
Step 2
Step 3
Step 4
Step 5
Step 6
Step 7
Step 8
Step 9
Step 10
Step 11

Step 5 – Rewrite the objective function with all the variables.

Maximize

$$Z = 100X_1 + 60X_2 + 40X_3 + 0S_1 + 0S_2 + 0S_3$$

Linear Programming Problem – Simplex Method

Simple Simplex Table

		C_j						
BV	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3
S_1								
S_2								
S_3								
$Z = 0$		Z_j						
		Δ_j						

$$\frac{X_B}{X_K}$$

Linear Programming Problem – Simplex Method

Simplex Method
Step 1
Step 2
Step 3
Step 4
Step 5
Step 6
Step 7
Step 8
Step 9
Step 10
Step 11

Step 6: Let us take

BV Simplex method starts at the origin, i.e., X_1 , X_2 & X_3 are equal to zero. These zero variables are called as non basic variables and the remaining variables are called basic variables.

C_B Coefficients of basic variables in revised objective function

X_B Right hand side values of constraint equations for the initial table only. Further they will be changed as per the iterations.

Linear Programming Problem – Simplex Method

Simplex Method		
Step 1	C_j	Coefficients of variables in revised objective function
Step 2		
Step 3	X_j	Coefficient of variables in constraint equations
Step 4		
Step 5	Z_j	$C_B X_j$
Step 6	Δ_j	$Z_j - C_j$
Step 7	Key column	The column with most negative Δ_j
Step 8		
Step 9		
Step 10	X_K	Corresponding row element in the key column, which are strictly non zero and positive.
Step 11		

Linear Programming Problem – Simplex Method

Simplex Method
Step 1
Step 2
Step 3
Step 4
Step 5
Step 6
Step 7
Step 8
Step 9
Step 10
Step 11

$$\frac{X_B}{X_K}$$

Ratio between the X_B and X_K values. Note that, non zero positive values of X_K are only to be considered, otherwise ignore the ratio.

Z

Value of the objective function, which is equal to $C_B X_B$

By following the above notations construct the initial simplex table

Linear Programming Problem – Simplex Method

Initial simplex table

		C_j	100	60	40	0	0	0	
BV	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	$\frac{X_B}{X_K}$
S_1	0	100	1	1	1	1	0	0	$\frac{100}{1} = 100$
S_2	0	600	10	4	5	0	1	0	$\frac{600}{10} = 60$
S_3	0	300	2	2	6	0	0	1	$\frac{300}{2} = 150$
$Z = 0$		Z_j	0	0	0	0	0	0	
		Δ_j	-100	-60	-40	0	0	0	

Linear Programming Problem – Simplex Method

Simplex Method
Step 1
Step 2
Step 3
Step 4
Step 5
Step 6
Step 7
Step 8
Step 9
Step 10
Step 11

Optimality test – If all the Δ_j values are positive or zero, then the solution is optimal, if any negative Δ_j appears, it means that at least one more iteration is required for the optimal solution.

Step 7 - If optimality condition passes, then stop. Value of Z at last tableau is the optimal value and the corresponding BV and X_B columns gives the variables and their values for the optimal allocation. Otherwise identify the entering variable for the next iteration by determining the column with most negative Δ_j value (or key column variable)

Linear Programming Problem – Simplex Method

Step 7: Identify the entering variable

		C_j	100	60	40	0	0	0
BV	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3
S_1	0	100	1	1	1	1	0	0
S_2	0	600	10	4	5	0	1	0
S_3	0	300	2	2	6	0	0	1
$Z = 0$		Z_j	0	0	0	0	0	0
		Δ_j	-100	-60	-40	0	0	0

$$\frac{X_B}{X_K}$$

$$\frac{100}{1} = 100$$

$$\frac{600}{10} = 60$$

$$\frac{300}{2} = 150$$

Linear Programming Problem – Simplex Method

Simplex Method
Step 1
Step 2
Step 3
Step 4
Step 5
Step 6
Step 7
Step 8
Step 9
Step 10
Step 11

Feasibility test – For the next iteration (if required) the leaving variable is the basic variable associated with the smallest ratio $\frac{X_B}{X_K}$.

Step 8 – Identify the leaving variable using the feasibility condition. The corresponding row of the leaving variable is called as the key row. The cell at the intersection of key row and key column is called as the key cell or key element.

Linear Programming Problem – Simplex Method

Step 8: Identify the leaving variable

		C_j	100	60	40	0	0	0	$\frac{X_B}{X_K}$
BV	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	
S_1	0	100	1	1	1	1	0	0	$\frac{100}{1} = 100$
S_2	0	600	10	4	5	0	1	0	$\frac{600}{10} = 60$
S_3	0	300	2	2	6	0	0	1	$\frac{300}{2} = 150$
$Z = 0$		Z_j	0	0	0	0	0	0	
		Δ_j	-100	-60	-40	0	0	0	

Linear Programming Problem – Simplex Method

Simplex Method
Step 1
Step 2
Step 3
Step 4
Step 5
Step 6
Step 7
Step 8
Step 9
Step 10
Step 11

Step 9 – Rewrite the table by making key cell value into 1 (unity) and the remaining key column values into 0 (zero).

Step 10 – While making other key column cell values into zeros, use newly formed key row in every calculation.

Note – In preparing the revised table, start with newly entered variable row by making it into one with appropriate multiplication factor

Step 11 – Repeat steps from 7 to 10 until all the Δ_j values are positive

Linear Programming Problem – Simplex Method

Iteration 1

		C_j	100	60	40	0	0	0	$\frac{X_B}{X_K}$
	BV	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3
$R_1' = R_1 - R_2'$	S_1	0	40	0	$\frac{3}{5}$	$\frac{1}{2}$	1	$-\frac{1}{10}$	$\frac{200}{3} = 66.7$
$R_2' = \frac{R_2}{10}$	X_1	100	60	1	$\frac{2}{5}$	$\frac{1}{2}$	0	$\frac{1}{10}$	$\frac{300}{0} = 150$
$R_3' = R_3 - 2R_2'$	S_3	0	180	0	$\frac{6}{5}$	5	0	$-\frac{1}{5}$	$\frac{900}{6} = 150$
$Z = 6000$		Z_j	100	40	50	0	10	0	
		Δ_j	0	-20	10	0	10	0	

Since all the Δ_j values are not positive, further iterations are needed to obtain the optimal solution.

Linear Programming Problem – Simplex Method

Iteration 2

$$R'_1 = \frac{5}{3}R_1$$

$$R'_2 = R_2 - \frac{2}{5}R'_1$$

$$R'_3 = R_3 - \frac{6}{5}R'_1$$

		C_j	100	60	40	0	0	0
BV	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3
X_2	60	$\frac{200}{3}$	0	1	$\frac{5}{6}$	$\frac{5}{3}$	$-\frac{1}{6}$	0
X_1	100	$\frac{100}{3}$	1	0	$\frac{1}{6}$	$-\frac{2}{3}$	$\frac{1}{6}$	0
S_3	0	100	0	0	4	-2	0	1
$Z = \frac{22000}{3}$		Z_j	100	60	$\frac{200}{3}$	$\frac{100}{3}$	$\frac{20}{3}$	0
		Δ_j	0	0	$\frac{80}{3}$	$\frac{100}{3}$	$\frac{20}{3}$	0

$\frac{X_B}{X_K}$

Since all the Δ_j values are positive or zero, this tableau is optimal.

Linear Programming Problem – Simplex Method

The optimal solution can be read from the last simplex tableau in the following manner.

The optimum value of the variables in the basic variable column are given in the X_B column and are as follows

Decision variable	Optimum value
X_1	$\frac{100}{3}$
X_2	$\frac{200}{3}$
X_3	0
Z	$\frac{22000}{3}$

APPENDIX

Slack Variable

$$6X_1 + 4X_2 \leq 24$$

Usage by activities

Limit on resource

- The difference between the right hand side and left hand side of the $\leq$ constraint yields the unused or slack amount of the resource.
- To convert $\leq$ inequality to an equation, a non negative slack variable is added to the left hand side of the constraint.

$$6X_1 + 4X_2 + S_1 = 24$$

APPENDIX

Surplus Variable

$$3X_1 + 5X_2 \geq 32$$

Usage by activities

Limit on resource

- The difference between the left hand side and the right hand side of the $\geq$ constraint represents surplus.
- To convert $\geq$ constraint into an equality, a non negative surplus variable is subtracted from the left hand side of the inequality.

$$3X_1 + 5X_2 - S_1 = 32$$

OPERATIONS RESEARCH

UNIT 1

Lecture 4



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Lecture Details:

Solution of Linear Programming Problem – Artificial Variable Technique – Big M Method

Operations Research/ME, 5 Semester.

LEARNING OUTCOMES

Upon completion of this lecture, student should be able to

- Explain the need for introduction of Artificial variable into the LP formulation
- Apply Big M method to determine the optimal solution to the given linear programming problem

Linear Programming (LP)

Artificial Variable Techniques

- Simplex method can be conveniently applied to linear programming problems of the standard form (Maximize Z subjected to “less than or equal to” constraints and non negativity constraint on all variables) by introducing the slack variables.
- Artificial variable techniques are used to solve linear programming problem of form other than standard form i.e., “greater than or equal to” or “equal to” constraints and negative right hand sides.
- This technique constructs a more convenient problem by introducing a artificial variable in the constraint equation, which needs one.

Linear Programming (LP)

Artificial Variable Techniques

- Artificial variables are used just to act like an initial basic variable for the constraint equation in the other than standard form.
- Objective function is modified by using a very large number as penalty of assigning artificial variables, a value higher than zero.
- The iterations of simplex method then automatically forces the artificial variables to disappear, one at a time, until they all are gone.

Linear Programming (LP)

Artificial Variable Techniques

Big - M (Charne's Penalty) Method - Example

Solve the following linear programming problem using Big M method

Minimize $Z = 3X_1 + 4X_2$

Subject to

$$2X_1 + 3X_2 \geq 8$$

$$5X_1 + 2X_2 \geq 12$$

$$X_1, X_2 \geq 0$$

Linear Programming (LP)

Artificial Variable Techniques

Big M Method - Example

Introducing surplus and artificial variables to convert inequality constraints into equations and rewriting the objective function, we get

$$\text{Minimize } Z = 3X_1 + 4X_2 + 0S_1 + 0S_2 + MA_1 + MA_2$$

Subject to

$$2X_1 + 3X_2 - S_1 + A_1 = 8$$

$$5X_1 + 2X_2 - S_2 + A_2 = 12$$

$$X_1, X_2, S_1, S_2, A_1, A_2 \geq 0$$

Linear Programming (LP)

Artificial Variable Techniques

Big M Method - Example

Converting minimization problem into maximization problem, we get

$$\text{Maximize } -Z = Z' = -3X_1 - 4X_2 + 0S_1 + 0S_2 - MA_1 - MA_2$$

Subject to

$$2X_1 + 3X_2 - S_1 + A_1 = 8$$

$$5X_1 + 2X_2 - S_2 + A_2 = 12$$

$$X_1, X_2, S_1, S_2, A_1, A_2 \geq 0$$

Linear Programming (LP)

Artificial Variable Techniques

Big M Method - Example

Iteration 0

		C_j	-3	-4	0	0	-M	-M
BV	C_B	X_B	X_1	X_2	S_1	S_2	A_1	A_2
A_1	-M	8	2	3	-1	0	1	0
A_2	-M	12	5	2	0	-1	0	1
$Z' = -20M$		Z_j	-7M	-5M	M	M	-M	-M
		Δ_j	3-7M	4-5M	M	M	0	0

$$\frac{X_B}{X_K}$$

$$\frac{8}{2} = 4$$

$$\frac{12}{5} = 2.4$$

Linear Programming (LP)

Artificial Variable Techniques

Big M Method - Example

Iteration 1

$$R'_1 = R_1 - 2R'_2$$

$$R'_2 = \frac{R_2}{5}$$

		C_j	-3	-4	0	0	-M	-M	
BV	C_B	X_B	X_1	X_2	S_1	S_2	A_1	A_2	
A_1	-M	$\frac{16}{5}$	0	$\frac{11}{5}$	-1	$\frac{2}{5}$	1	$-\frac{2}{5}$	$\frac{X_B}{X_K}$
X_1	-3	$\frac{12}{5}$	1	$\frac{2}{5}$	0	$-\frac{1}{5}$	0	$\frac{1}{5}$	$\frac{16}{11} = 1.4$
Z' $= -\frac{36 + 16M}{5}$		Z_j	-3	$-\frac{6 - 11M}{5}$	M	$\frac{3 - 2M}{5}$	-M	$-\frac{3 + 2M}{5}$	$\frac{12}{2} = 6$
		Δ_j	0	$\frac{14 - 11M}{5}$	M	$\frac{3 - 2M}{5}$	0	$\frac{7M - 3}{5}$	
				$\frac{14 - 11M}{5}$		$\frac{3 - 2M}{5}$		$\frac{7M - 3}{5}$	
				$\frac{14 - 11M}{5}$		$\frac{3 - 2M}{5}$		$\frac{7M - 3}{5}$	

Linear Programming (LP)

Artificial Variable Techniques

Big M Method - Example

Iteration 2

$$R'_1 = R_1 \times \frac{5}{11}$$

$$R'_2 = R_2 - \frac{2}{5}R_1$$

		C_j	-3	-4	0	0	-M	-M
BV	C_B	X_B	X_1	X_2	S_1	S_2	A_1	A_2
X_2	-4	$\frac{16}{11}$	0	1	$-\frac{5}{11}$	$\frac{2}{11}$	$\frac{5}{11}$	$-\frac{2}{11}$
X_1	-3	$\frac{100}{55}$	1	0	$\frac{2}{11}$	$-\frac{15}{55}$	$-\frac{2}{11}$	$\frac{15}{55}$
$Z' = -\frac{124}{11}$		Z_j	-3	-4	$\frac{14}{11}$	$\frac{1}{11}$	$-\frac{14}{11}$	$-\frac{17}{11}$
		Δ_j	0	0	$\frac{14}{11}$	$\frac{1}{11}$	$M - \frac{14}{11}$	$M - \frac{17}{11}$

$\frac{X_B}{X_K}$

Since all the Δ_j values are either positive or zero, this simplex tableau is optimal.

Linear Programming (LP)

Artificial Variable Techniques

Big M Method - Example

From the last simplex tableau, the optimal solution is,

Decision variable	Optimum value
X_1	$\frac{100}{55}$
X_2	$\frac{16}{11}$
Z	$\frac{124}{11}$

Note: From last simplex tableau, we get $Z' = -\frac{124}{11}$, since $Z' = -Z$, we get, $Z = \frac{124}{11}$ which is the minimum value of the objective function

OPERATIONS RESEARCH

UNIT 1 Lecture 5



Presented By:

Mr. K. Sreenivasa Reddy

Assistant Professor

Mechanical Engineering

GIET(A)

Lecture Details:

Solution of Linear Programming Problem – Artificial Variable Technique – Two Phase Method

Operations Research/ME, 5 Semester.

LEARNING OUTCOMES

Upon completion of this lecture, student should be able to

- Explain the limitation of Big M method for problems involving artificial variables
- Apply two phase method to determine the optimal solution to the given linear programming problem

Linear Programming (LP)

Artificial Variable Techniques

Two Phase Method

- Accuracy of the computations of simplex method are impaired by the large penalty imposed in objective function on the artificial variables for having other than zero value.
- This problem can be rectified by the two phase method by eliminating the constant M altogether.
- This technique solves the linear programming problem in two phases. Phase I attempts to find a starting basic feasible solution, and if one is found, phase II is used to solve the original problem.

Linear Programming (LP)

Artificial Variable Techniques

Two Phase Method - Example

Use the two phase method to solve the following linear programming problem.

Minimize $Z = 0.4X_1 + 0.5X_2$

Subject to

$$0.3X_1 + 0.1X_2 \leq 2.7$$

$$0.5X_1 + 0.5X_2 = 6$$

$$0.6X_1 + 0.4X_2 \geq 6$$

$$X_1, X_2 \geq 0$$

Linear Programming (LP)

Artificial Variable Techniques

Two Phase Method - Example

Phase I problem

Minimize $Z = A_1 + A_2$

Subject to

$$0.3X_1 + 0.1X_2 + S_1 = 2.7$$

$$0.5X_1 + 0.5X_2 + A_1 = 6$$

$$0.6X_1 + 0.4X_2 - S_2 + A_2 = 6$$

$$X_1, X_2, S_1, S_2, A_1, A_2 \geq 0$$

Linear Programming (LP)

Artificial Variable Techniques

Two Phase Method - Example

Phase I problem

Maximize $-Z = -A_1 - A_2$

Subject to

$$0.3X_1 + 0.1X_2 + S_1 = 2.7$$

$$0.5X_1 + 0.5X_2 + A_1 = 6$$

$$0.6X_1 + 0.4X_2 - S_2 + A_2 = 6$$

$$X_1, X_2, S_1, S_2, A_1, A_2 \geq 0$$

Linear Programming (LP)

Artificial Variable Techniques

Two Phase Method - Example

Iteration 0

		C_j	0	0	0	0	-1	-1
BV	C_B	X_B	X_1	X_2	S_1	S_2	A_1	A_2
S_1	0	2.7	0.3	0.1	1	0	0	0
A_1	-1	6	0.5	0.5	0	0	1	0
A_2	-1	6	0.6	0.4	0	-1	0	1
$Z' = -12$		Z_j	-1.1	-0.9	0	1	-1	-1
		Δ_j	-1.1	-0.9	0	1	0	0

$$\frac{X_B}{X_K}$$

$$\frac{2.7}{0.3} = 9$$

$$\frac{6}{0.5} = 12$$

$$\frac{6}{0.6} = 10$$

Linear Programming (LP)

Artificial Variable Techniques

Two Phase Method - Example

Iteration 1

$$R'_1 = \frac{R_1}{0.3}$$

$$R'_2 = R_2 - 0.5R'_1$$

$$R'_3 = R_3 - 0.6R'_1$$

		C_j	0	0	0	0	-1	-1
BV	C_B	X_B	X_1	X_2	S_1	S_2	A_1	A_2
X_1	0	9	1	0.33	3.33	0	0	0
A_1	-1	1.5	0	0.33	-1.67	0	1	0
A_2	-1	0.6	0	0.2	-2	-1	0	1
$Z' = -2.1$		Z_j	0	-0.53	3.67	1	-1	-1
		Δ_j	0	-0.53	3.67	1	0	0

$$\frac{X_B}{X_K}$$

$$\frac{9}{0.33} = 27.3$$

$$\frac{1.5}{0.33} = 4.54$$

$$\frac{0.6}{0.2} = 3$$

Linear Programming (LP)

Artificial Variable Techniques

Two Phase Method - Example

Iteration 2

$$R'_1 = R_1 - 0.33R'_3$$

$$R'_2 = R_2 - 0.33R'_3$$

$$R'_3 = \frac{R_3}{0.2}$$

		C_j	0	0	0	0	-1	-1
BV	C_B	X_B	X_1	X_2	S_1	S_2	A_1	A_2
X_1	0	8	1	0	6.63	1.65	0	-1.65
A_1	-1	0.5	0	0	1.63	1.65	1	-1.65
X_2	0	3	0	1	-10	-5	0	5
$Z' = -0.5$		Z_j	0	0	-1.63	-1.65	-1	1.65
		Δ_j	0	0	-1.63	-1.65	0	2.65

$$\frac{X_B}{X_K}$$

$$\frac{8}{1.65} = 4.8$$

$$\frac{0.5}{1.65} = 0.3$$

Ignore

Linear Programming (LP)

Artificial Variable Techniques

Two Phase Method - Example

Iteration 3

$$R'_1 = R_1 - 1.65R'_2$$

$$R'_2 = \frac{R_2}{1.65}$$

$$R'_3 = R_3 + 5R'_2$$

		C_j	0	0	0	0	-1	-1
BV	C_B	X_B	X_1	X_2	S_1	S_2	A_1	A_2
X_1	0	7.5	1	0	5	0	-1	0
S_2	0	0.3	0	0	0.99	1	0.61	-1
X_2	0	4.5	0	1	-5.1	0	3	0
$Z' = 0$		Z_j	0	0	0	0	0	0
		Δ_j	0	0	0	0	1	1

$$\frac{X_B}{X_K}$$

Since there are no artificial variables in basic variable column, this represents a feasible basic solutions, and is the starting table for phase II with slight modification

Linear Programming (LP)

Artificial Variable Techniques

Two Phase Method - Example

Phase II problem

Maximize

$$-Z = -0.4X_1 - 0.5X_2$$

Subject to

$$0.3X_1 + 0.1X_2 + S_1 = 2.7$$

$$0.5X_1 + 0.5X_2 = 6$$

$$0.6X_1 + 0.4X_2 - S_2 = 6$$

$$X_1, X_2, S_1, S_2 \geq 0$$

Linear Programming (LP)

Artificial Variable Techniques

Two Phase Method - Example

Iteration 0

		C_j	-0.4	-0.5	0	0
BV	C_B	X_B	X_1	X_2	S_1	S_2
X_1	-0.4	7.5	1	0	5	0
S_2	0	0.3	0	0	0.99	1
X_2	-0.5	4.5	0	1	-5.1	0
$Z' = -5.25$		Z_j	-0.4	-0.5	0.55	0
		Δ_j	0	0	0.55	0

$$\frac{X_B}{X_K}$$

Since all the Δ_j values are either positive or zero, this is the optimal simplex tableau.

Linear Programming (LP)

Artificial Variable Techniques

Two Phase Method - Example

From the last simplex tableau, the optimal solution is,

Decision variable	Optimal solution
X_1	7.5
X_2	4.5
Z	5.25

OPERATIONS RESEARCH

UNIT 1 Lecture 6



Presented By:

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GIET(A)

Lecture Details:

**Solution of Linear Programming Problem – Multiple
Optimal Solutions, Degeneracy**

Operations Research/ME, 5 Semester.

LEARNING OUTCOMES

Upon completion of this lecture, student should be able to

- Understand the special cases of multiple optimal solutions and degeneracy in solution of linear programming problems.
- Illustrate the cases of multiple optimal solutions degeneracy in solution of linear programming problems.

Linear Programming (LP)

Multiple Optimal Solutions

Numerical Example

- A linear programming problem can have more than one optimal solution (occurs when the objective function is parallel to the constraint equation). In this case, the objective function has same value at more than one corner point.
- Whenever the linear programming problem has alternate optimum, at least one of the non basic variables has zero coefficient in the Z row. This means, increasing such variable will not change the objective function value.
- By choosing the non basic variable with zero coefficient as entering variable in extra iteration after reaching optimum solution for first time, alternate optimum solutions can be determined.

Linear Programming (LP)

Multiple Optimal Solutions

Numerical Example

Solve the following linear programming problem

Maximize $Z = 2X_1 + 4X_2$

Subject to

$$X_1 + 2X_2 \leq 5$$

$$X_1 + X_2 \leq 4$$

$$X_1, X_2 \geq 0$$

Linear Programming (LP)

Multiple Optimal Solutions

Numerical Example

Iteration 0

		C_j	2	4	0	0
BV	C_B	X_B	X_1	X_2	S_1	S_2
S_1	0	5	1	2	1	0
S_2	0	4	1	1	0	1
$Z = 0$		Z_j	0	0	0	0
		Δ_j	-2	-4	0	0

$$\frac{X_B}{X_K}$$

$$\frac{5}{2} = 2.5$$

$$\frac{4}{1} = 4$$

Linear Programming (LP)

Multiple Optimal Solutions

Numerical Example

Iteration 1

$$R'_1 = \frac{R_1}{2}$$

$$R'_2 = R_2 - R'_1$$

		C_j	2	4	0	0
BV	C_B	X_B	X_1	X_2	S_1	S_2
X_2	4	2.5	0.5	1	0.5	0
S_2	0	1.5	0.5	0	-0.5	1
$Z = 10$		Z_j	2	4	2	0
		Δ_j	0	0	2	0

$$\frac{X_B}{X_K}$$

Since all the Δ_j values are either positive or zero, this simplex tableau is optimal. But as the non basic variable, X_1 has zero Δ_j value, it represents the existence of alternate optimum. The alternative optimal solution can be determined by performing one more iteration by taking X_1 as entering variable.

Linear Programming (LP)

Multiple Optimal Solutions

Numerical Example

Iteration 1

$$R'_1 = \frac{R_1}{2}$$

$$R'_2 = R_2 - R'_1$$

		C_j	2	4	0	0
BV	C_B	X_B	X_1	X_2	S_1	S_2
X_2	4	2.5	0.5	1	0.5	0
S_2	0	1.5	0.5	0	-0.5	1
$Z = 10$		Z_j	2	4	2	0
		Δ_j	0	0	2	0

$$\frac{X_B}{X_K}$$

$$\frac{2.5}{0.5} = 5$$

$$\frac{1.5}{0.5} = 3$$

Linear Programming (LP)

Multiple Optimal Solutions

Numerical Example

Iteration 2

$$R'_1 = R_1 - 0.5R'_2$$

$$R'_2 = \frac{R_2}{0.5}$$

		C_j	2	4	0	0
BV	C_B	X_B	X_1	X_2	S_1	S_2
X_2	4	1	0	1	1	-1
X_1	2	3	1	0	-1	2
$Z = 10$		Z_j	2	4	2	0
		Δ_j	0	0	2	0

$$\frac{X_B}{X_K}$$

$$\frac{2.5}{0.5} = 5$$

$$\frac{1.5}{0.5} = 3$$

Since all the Δ_j values are either positive or zero, this simplex tableau is optimal.

Linear Programming (LP)

Multiple Optimal Solutions

Numerical Example

The multiple optimal solutions for the given problem are

Decision variable	Optimum value
X_1	0
X_2	2.5
Z	10

or

Decision variable	Optimum value
X_1	3
X_2	1
Z	10

Linear Programming (LP) Degeneracy

- Whenever tie for the minimum ratio occurs, at least one basic variable will be zero in the next basic feasible solution (basic variables with zero value are called degenerate).
- Whenever this occurs, the value of objective function remains unchanged and the simplex method may then go around in a loop.
- It rarely occurs in practical problems. Special rules are constructed to break the loop, but they make the computation slower, hence are omitted in many softwares for LP.

Linear Programming (LP) Degeneracy Numerical Example 1

Solve the following linear programming problem

$$\text{Maximize } Z = 2X_1 + X_2$$

Subject to

$$4X_1 + 3X_2 \leq 12$$

$$4X_1 + X_2 \leq 8$$

$$4X_1 - X_2 \leq 8$$

$$X_1, X_2 \geq 0$$

Linear Programming (LP) Degeneracy Numerical Example 1

Converting inequality constraints into equations, we get

$$\text{Maximize } Z = 2X_1 + X_2 + 0S_1 + 0S_2 + 0S_3$$

Subject to

$$4X_1 + 3X_2 + S_1 = 12$$

$$4X_1 + X_2 + S_2 = 8$$

$$4X_1 - X_2 + S_3 = 8$$

$$X_1, X_2, S_1, S_2, S_3 \geq 0$$

Linear Programming (LP)

Degeneracy

Numerical Example 1

Iteration 0

		C_j	2	1	0	0	0
BV	C_B	X_B	X_1	X_2	S_1	S_2	S_3
S_1	0	12	4	3	1	0	0
S_2	0	8	4	1	0	1	0
S_3	0	8	4	-1	0	0	1
$Z = 0$		Z_j	0	0	0	0	0
		Δ_j	-2	-1	0	0	0

$$\frac{X_B}{X_K}$$

$$\frac{12}{4} = 3$$

$$\frac{8}{4} = 2$$

$$\frac{8}{4} = 2$$

There is a tie among the basic variables for minimum ratio. Choosing S_3 as the leaving variable arbitrarily, we get

Linear Programming (LP)

Degeneracy

Numerical Example 1

Iteration 1

$$R'_1 = R_1 - 4R'_3$$

$$R'_2 = R_2 - 4R'_3$$

$$R'_3 = \frac{R_3}{4}$$

		C_j	2	1	0	0	0
BV	C_B	X_B	X_1	X_2	S_1	S_2	S_3
S_1	0	4	0	4	1	0	-1
S_2	0	0	0	2	0	1	-1
X_1	2	2	1	-0.25	0	0	0.25
$Z = 4$		Z_j	2	-0.5	0	0	0.5
		Δ_j	0	-1.5	0	0	0.5

$$\frac{X_B}{X_K}$$

$$\frac{4}{4} = 1$$

0

Ignore

Linear Programming (LP) Degeneracy Numerical Example 1

Iteration 2

$$R'_1 = R_1 - 4R'_2$$

$$R'_2 = \frac{R_2}{2}$$

$$R'_3 = R_3 + 0.25R'_2$$

		C_j	2	1	0	0	0	
BV	C_B	X_B	X_1	X_2	S_1	S_2	S_3	$\frac{X_B}{X_K}$
S_1	0	4	0	0	1	-2	1	$\frac{4}{1} = 4$
X_2	1	0	0	1	0	0.5	-0.5	Ignore
X_1	2	2	1	0	0	0.13	0.13	$\frac{2}{0.13} = 15$
$Z = 4$		Z_j	2	1	0	0.76	-0.24	
		Δ_j	0	0	0	0.76	-0.24	

For iterations 1 and 2, the objective function did not increase, and the basic variable X_2 is zero, which is the indication of degeneracy

Linear Programming (LP) Degeneracy Numerical Example 1

Iteration 3

$$R'_1 = \frac{R_1}{1}$$

$$R'_2 = R_1 + 0.5R'_1$$

$$R'_3 = R_3 - 0.13R'_1$$

		C_j	2	1	0	0	0
BV	C_B	X_B	X_1	X_2	S_1	S_2	S_3
S_3	0	4	0	0	1	-2	1
X_2	1	2	0	1	0.5	-0.5	0
X_1	2	1.5	1	0	-0.13	0.39	0
$Z = 5$		Z_j	2	1	0.24	0.28	0
		Δ_j	0	0	0.24	0.28	0

$\frac{X_B}{X_K}$

This simplex tableau is optimal and the optimum solution is $X_1 = 1.5$ and $X_2 = 2$ with $Z = 5$ and is obtained in 3 iterations, with one degenerate iteration.

Linear Programming (LP)

Degeneracy

Numerical Example 1

Iteration 0

		C_j	2	1	0	0	0
BV	C_B	X_B	X_1	X_2	S_1	S_2	S_3
S_1	0	12	4	3	1	0	0
S_2	0	8	4	1	0	1	0
S_3	0	8	4	-1	0	0	1
$Z = 0$		Z_j	0	0	0	0	0
		Δ_j	-2	-1	0	0	0

$$\frac{X_B}{X_K}$$

$$\frac{12}{4} = 3$$

$$\frac{8}{4} = 2$$

$$\frac{8}{4} = 2$$

Now, choosing S_2 to be the leaving variable instead of S_3 , we get

Linear Programming (LP) Degeneracy Numerical Example 1

Iteration 1

$$R'_1 = R_1 - 4R'_2$$

$$R'_2 = \frac{R_2}{4}$$

$$R'_3 = R_3 - 4R'_2$$

		C_j	2	1	0	0	0
BV	C_B	X_B	X_1	X_2	S_1	S_2	S_3
S_1	0	4	0	2	1	-1	0
X_1	2	2	1	0.25	0	0.25	0
S_3	0	0	0	-2	0	-1	1
$Z = 4$		Z_j	2	0.5	0	0.5	0
		Δ_j	0	-0.5	0	0.5	0

$$\frac{X_B}{X_K}$$

$$\frac{4}{2} = 2$$

$$\frac{2}{0.25} = 8$$

Ignore

Linear Programming (LP)

Degeneracy

Numerical Example 1

Iteration 2

$$R'_1 = \frac{R_1}{2}$$

$$R'_2 = R_2 - 0.25R'_1$$

$$R'_3 = R_3 + 2R'_1$$

		C_j	2	1	0	0	0
BV	C_B	X_B	X_1	X_2	S_1	S_2	S_3
X_2	1	2	0	1	0.5	-0.5	0
X_1	2	1.5	1	0	-0.13	0.38	0
S_3	0	4	0	0	1	-2	1
$Z = 5$		Z_j	2	1	0.24	0.26	0
		Δ_j	0	0	0.24	0.26	0

$$\frac{X_B}{X_K}$$

$$\frac{4}{2} = 2$$

$$\frac{2}{0.25} = 8$$

Ignore

This simplex tableau is optimal, and the optimal solution is $X_1 = 1.5$ and $X_2 = 2$ with $Z = 5$ and is obtained in 2 iterations. One rule to handle degeneracy is to select variable with smaller subscript to break the tie for minimum ratio.

OPERATIONS RESEARCH

UNIT 1 Lecture 7



Presented By:

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Lecture Details:

Solution of Linear Programming Problem – Unbounded Solution & Infeasible Solution

Operations Research/ME, 5 Semester.

LEARNING OUTCOMES

Upon completion of this lecture, student should be able to

- Understand the occurrence of unbounded, infeasible solutions in the solution of linear programming problems.
- Illustrate the cases of unbounded and infeasible solutions of linear programming problems.

Linear Programming (LP)

Unbounded Solution

- In some LP problems, the values of the decision variables may be increased indefinitely without violating any of the constraints. This leads to unbounded solution space.
- The objective function may be increased (decreased) in maximization (minimization) problem indefinitely.
- This may occur when one or more non redundant constraints have not taken into consideration or parameters for constraints are not estimated correctly.
- In simplex table, unbounded solution occurs when no variable qualifies as leaving variable.

Linear Programming (LP)

Unbounded Solution

Numerical Example

Solve the following linear programming problem

Maximize $Z = 20X_1 + 10X_2 + X_3$

Subject to

$$3X_1 - 3X_2 + 5X_3 \leq 50$$

$$X_1 + X_3 \leq 10$$

$$X_1 - X_2 + 4X_3 \leq 20$$

$$X_1, X_2, X_3 \geq 0$$

Linear Programming (LP)

Unbounded Solution

Numerical Example

Converting inequality constraints into equations, we get

Maximize $Z = 20X_1 + 10X_2 + X_3 + 0S_1 + 0S_2 + 0S_3$

Subject to

$$3X_1 - 3X_2 + 5X_3 + S_1 = 50$$

$$X_1 + X_3 + S_2 = 10$$

$$X_1 - X_2 + 4X_3 + S_3 = 20$$

$$X_1, X_2, X_3 \geq 0$$

Linear Programming (LP)

Unbounded Solution

Numerical Example

Iteration 0

		C_j	20	10	1	0	0	0
BV	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3
S_1	0	50	3	-3	5	1	0	0
S_2	0	10	1	0	1	0	1	0
S_3	0	20	1	-1	4	0	0	1
$Z = 0$		Z_j	0	0	0	0	0	0
		Δ_j	-20	-10	-1	0	0	0

$$\frac{X_B}{X_K}$$

$$\frac{50}{3} = 16.7$$

$$\frac{10}{1} = 10$$

$$\frac{20}{1} = 20$$

Linear Programming (LP)

Unbounded Solution

Numerical Example

Iteration 1

$$R_1 = R_1 - 3R_2'$$

$$R_2' = \frac{R_2}{1}$$

$$R_3' = R_3 - R_2'$$

		C_j	20	10	1	0	0	0	
BV	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	$\frac{X_B}{X_K}$
S_1	0	20	0	-3	2	1	-3	0	Ignore
X_1	20	10	1	0	1	0	1	0	Ignore
S_3	0	10	0	-1	3	0	-1	1	Ignore
$Z = 200$	Z_j	20	0	0	20	0	20	0	
	Δ_j	0	-10	19	0	20	0	0	

There is no variable, which qualifies to be the leaving variable. This indicates that the entering variable X_2 can be increased indefinitely or the objective function Z is unbounded.

Linear Programming (LP)

Infeasible Solution

- Linear programming problems with inconsistent constraints have no feasible solution.
- This case never occurs if all the constraints are “less than or equal to” type with non negative right hand sides.
- If the problem has infeasible solution, then at least one artificial variable will be present in the optimum iteration.
- The infeasibility may occurs when the linear programming problem is not formulated correctly.

Linear Programming (LP)

Infeasible Solution

Numerical Example

Solve the following linear programming problem

Maximize $Z = 3X_1 + 2X_2$

Subject to

$$2X_1 + X_2 \leq 2$$

$$3X_1 + 4X_2 \geq 12$$

$$X_1, X_2 \geq 0$$

Linear Programming (LP)

Infeasible Solution

Numerical Example

Converting the inequality constraints into equations, we get

Maximize $Z = 3X_1 + 2X_2 + 0S_1 + 0S_2 - MA_1$

Subject to

$$2X_1 + X_2 + S_1 = 2$$

$$3X_1 + 4X_2 - S_2 + A_1 = 12$$

$$X_1, X_2, S_1, S_2, A_1 \geq 0$$

Linear Programming (LP)

Infeasible Solution

Numerical Example

Iteration 0

		C_j	3	2	0	-M	0
BV	C_B	X_B	X_1	X_2	S_1	A_1	S_2
S_1	0	2	2	1	1	0	0
A_1	-M	12	3	4	0	1	-1
$Z = -12M$		Z_j	-3M	-4M	0	-M	M
		Δ_j	-3-3M	-2-4M	0	0	M

$$\frac{X_B}{X_K}$$

$$\frac{2}{1} = 2$$

$$\frac{12}{4} = 3$$

Linear Programming (LP)

Infeasible Solution

Numerical Example

Iteration 1

$$R'_1 = \frac{R_1}{1}$$

$$R'_2 = R_2 - 4R'_1$$

		C_j	3	2	0	-M	0
BV	C_B	X_B	X_1	X_2	S_1	A_1	S_2
X_2	2	2	2	1	1	0	0
A_1	-M	4	-5	0	-4	1	-1
$Z = 4 - 4M$		Z_j	4+5M	2	2+4M	-M	M
		Δ_j	5M+1	0	4M+2	0	M

$\frac{X_B}{X_K}$

Optimum iteration shows that the artificial variable A_1 is positive, which indicated that the problem is infeasible.

OPERATIONS RESEARCH

UNIT 1 Lecture 8



Presented By:

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Lecture Details:

Duality Theory

Operations Research/ME, 5 Semester.

LEARNING OUTCOMES

Upon completion of this lecture, student should be able to

- Explain the concept of duality and primal – dual relationship
- Solve a linear programming problem by converting it into its dual

Linear Programming (LP)

Duality Theory

Solve the following linear programming problem

Maximize $Z = 3X_1 + 5X_2$

Subject to

$$1X_1 + 0X_2 \leq 4$$

$$0X_1 + 2X_2 \leq 12$$

$$3X_1 + 2X_2 \leq 18$$

$$X_1, X_2 \geq 0$$

Maximize $Z = c_1X_1 + c_2X_2$

Subject to

$$a_{11}X_1 + a_{12}X_2 \leq b_1$$

$$a_{21}X_1 + a_{22}X_2 \leq b_2$$

$$a_{31}X_1 + a_{32}X_2 \leq b_3$$

$$X_1, X_2 \geq 0$$

Linear Programming (LP)

Duality Theory

Converting inequality constraints into equations, we get

Maximize $Z = 3X_1 + 5X_2 + 0S_1 + 0S_2 + 0S_3$

Subject to

$$1X_1 + 0X_2 + S_1 = 4$$

$$0X_1 + 2X_2 + S_2 = 12$$

$$3X_1 + 2X_2 + S_3 = 18$$

$$X_1, X_2, S_1, S_2, S_3 \geq 0$$

Linear Programming (LP) Duality Theory

Iteration 0

		C_j	3	5	0	0	0
BV	C_B	X_B	X_1	X_2	S_1	S_2	S_3
S_1	0	4	1	0	1	0	0
S_2	0	12	0	2	0	1	0
S_3	0	18	3	2	0	0	1
$Z = 0$		Z_j	0	0	0	0	0
		Δ_j	-3	-5	0	0	0

$$\frac{X_B}{\bar{X}_K}$$

Ignore

$$\frac{12}{2} = 6$$

$$\frac{18}{2} = 9$$

Linear Programming (LP) Duality Theory

Iteration 1

$$R'_1 = R_1$$

$$R'_2 = \frac{R_2}{2}$$

$$R'_3 = R_3 - 2R'_2$$

		C_j	3	5	0	0	0
BV	C_B	X_B	X_1	X_2	S_1	S_2	S_3
S_1	0	4	1	0	1	0	0
X_2	5	6	0	1	0	$\frac{1}{2}$	0
S_3	0	6	3	0	0	-1	1
$Z = 30$		Z_j	0	5	0	2.5	0
		Δ_j	-3	0	0	2.5	0

$$\frac{X_B}{\bar{X}_K}$$

Ignore

$$\frac{12}{2} = 6$$

$$\frac{18}{2} = 9$$

Linear Programming (LP) Duality Theory

Iteration 1

$$R'_1 = R_1$$

$$R'_2 = \frac{R_2}{2}$$

$$R'_3 = R_3 - 2R'_2$$

		C_j	c_1	c_2	0	0	0
BV	C_B	X_B	X_1	X_2	S_1	S_2	S_3
S_1	0	4	1	0	1	0	0
X_2	5	6	0	1	0	$\frac{1}{2}$	0
S_3	0	6	3	0	0	-1	1
$W = 30$		Z_j	z_1	z_2	y_1	y_2	y_3
		Δ_j	$z_1 - c_1$	$z_2 - c_2$	y_1	y_2	y_3

$$\frac{X_B}{\bar{X}_K}$$

Ignore

$$\frac{12}{2} = 6$$

$$\frac{18}{2} = 9$$

Linear Programming (LP) Duality Theory

$W = 30$	Z_j	z_1	z_2	y_1	y_2	y_3
	Δ_j	$z_1 - c_1$	$z_2 - c_2$	y_1	y_2	y_3

$Z = 30$	Z_j	0	5	0	2.5	0
	Δ_j	-3	0	0	2.5	0

$$\begin{array}{|c|c|} \hline a_{11} & a_{12} \\ \hline a_{21} & a_{22} \\ \hline a_{31} & a_{32} \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & 2 \\ \hline 3 & 2 \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline b_1 \\ \hline b_2 \\ \hline b_3 \\ \hline \end{array} = \begin{array}{|c|} \hline 4 \\ \hline 12 \\ \hline 18 \\ \hline \end{array}$$

$$\begin{aligned} z_1 &= a_{11}y_1 + a_{21}y_2 + a_{31}y_3 \\ &= 1 \times 0 + 0 \times 2.5 + 3 \times 0 = 0 \end{aligned}$$

$$\begin{aligned} z_2 &= a_{12}y_1 + a_{22}y_2 + a_{32}y_3 \\ &= 0 \times 0 + 2 \times 2.5 + 2 \times 0 = 5 \end{aligned}$$

$$\begin{aligned} W &= y_1b_1 + y_2b_2 + y_3b_3 \\ &= 0 \times 4 + 2.5 \times 12 + 0 \times 18 = 30 \end{aligned}$$

Linear Programming (LP)

Duality Theory

$$z_1 = a_{11}y_1 + a_{21}y_2 + a_{31}y_3$$

$$z_2 = a_{12}y_1 + a_{22}y_2 + a_{32}y_3$$

$$W = y_1b_1 + y_2b_2 + y_3b_3$$

$$z_j = \sum_{i=1}^3 a_{ij} y_i \text{ for } j = 1, 2$$

$$W = \sum_{i=1}^3 b_i y_i$$

Linear Programming (LP) Duality Theory

Iteration 1

$$R'_1 = R_1$$

$$R'_2 = \frac{R_2}{2}$$

$$R'_3 = R_3 - 2R'_2$$

		C_j	c_1	c_2	0	0	0	
BV	C_B	X_B	X_1	X_2	S_1	S_2	S_3	$\frac{X_B}{X_K}$
S_1	0	4	1	0	1	0	0	Ignore
X_2	5	6	0	1	0	$\frac{1}{2}$	0	$\frac{12}{2} = 6$
S_3	0	6	3	0	0	-1	1	$\frac{18}{2} = 9$
$W = 30$		Z_j	z_1	z_2	y_1	y_2	y_3	
		Δ_j	$z_1 - c_1$	$z_2 - c_2$	y_1	y_2	y_3	

For optimum solution, all Δ_j should be non negative or at least zero, i.e., $z_j - c_j \geq 0$ and $y_i \geq 0$

Linear Programming (LP) Duality Theory

The conditions of optimality are $z_j - c_j \geq 0$ for $j = 1, 2$ and $y_i \geq 0$ for $i=1, 2, 3$

So the simplex method seeks values of y_i , such that

Maximize/Minimize
?

$$W = \sum_{i=1}^3 b_i y_i$$

$$\sum_{i=1}^3 a_{ij} y_i \geq c_j \quad \text{for } j = 1, 2$$

and

$$y_i \geq 0 \quad \text{for } i = 1, 2, 3$$

Looks like
LPP?

Linear Programming (LP) Duality Theory

Primal

$$\begin{aligned} & \text{Maximize} && Z = \sum_{j=1}^2 c_j X_j \\ & \text{Subject to} && \\ & && \sum_{j=1}^2 a_{ij} X_j \leq b_i \quad \text{for } i = 1, 2, 3 \\ & && \text{and} \\ & && X_j \geq 0 \quad \text{for } j = 1, 2 \end{aligned}$$

Dual

$$\begin{aligned} & \text{Minimize} && W = \sum_{i=1}^3 b_i y_i \\ & \text{Subject to} && \\ & && \sum_{i=1}^3 a_{ij} y_i \geq c_j \quad \text{for } j = 1, 2 \\ & && \text{and} \\ & && y_i \geq 0 \quad \text{for } i = 1, 2, 3 \end{aligned}$$

Linear Programming (LP)

Duality Theory

Primal

Dual

$$\begin{aligned}
 & \text{Maximize} \quad Z = \sum_{j=1}^n c_j X_j \\
 & \text{Subject to} \\
 & \sum_{j=1}^n a_{ij} X_j \leq b_i \quad \text{for } i = 1, 2, \dots, m \\
 & \text{and} \\
 & X_j \geq 0 \quad \text{for } j = 1, 2, \dots, n
 \end{aligned}$$

$$\begin{aligned}
 & \text{Minimize} \quad W = \sum_{i=1}^m b_i y_i \\
 & \text{Subject to} \\
 & \sum_{i=1}^m a_{ij} y_i \geq c_j \quad \text{for } j = 1, 2, \dots, n \\
 & \text{and} \\
 & y_i \geq 0 \quad \text{for } i = 1, 2, \dots, m
 \end{aligned}$$

Linear Programming (LP)

Duality Theory

Weak duality property – If x is a feasible solution for the primal problem and y is a feasible solution for the dual problem, then

$$cx \leq yb$$

Strong duality property – If x^* is an optimal solution for the primal problem and y^* is an optimal solution for the dual problem, then

$$cx^* = y^*b$$

Linear Programming (LP)

Duality Theory

Duality theorem

- If one problem has feasible solutions and a bounded objective function (and so has an optimal solution), then so does the other problem, so both the weak and strong duality properties are applicable
- If one problem has feasible solutions and an unbounded objective function (and so no optimal solution), then the other problem has no feasible solution.
- If one problem has no feasible solutions, then the other problem has either no feasible solution or an unbounded objective function.

Linear Programming (LP)

Duality Theory

Applications

- The number of functional constraints affects the computational effort of the simplex method far more than the number of variables does. If $m > n$, then applying the simplex method directly to the dual problem instead of the primal problem probably will achieve a substantial reduction in computational effort.
- If a feasible solution, y can be found for dual problem, and x can be found for primal problem by inspection, such that $cx = yb$, then x must be the optimal solution of primal problem, found without applying simplex method. Always it provides upper bound on optimal value of primal problem.
- Sensitivity analysis, insights on primal problem through economic interpretation of dual are some other applications.

Linear Programming (LP)

Dual Problem

- A dual variable is defined for each primal (constraint) equation.
- A dual constraint is defined for each primal variable.
- The constraint coefficients of the primal variable define the left hand side coefficients of the dual constraint and its objective coefficient define the right hand side.
- The objective coefficients of the dual equal the right hand side of the primal constraint equations.

Linear Programming (LP)

Dual Problem

Rules for constructing the dual problem

Primal problem (or dual problem)	Dual problem (or primal problem)
Maximize Z (or W)	Minimize W (or Z)
Constraint i:	Variable y_i (or x_i):
$\leq$	$y_i \geq 0$
$=$	Unconstrained
$\geq$	$y_i' \leq 0$
Variable x_j (or y_i):	Constraint j:
$x_j \geq 0$	$\geq$
Unconstrained	$=$
$x_j' \leq 0$	$\leq$

Linear Programming (LP)

Dual Problem – Example 1

Write the dual of the following linear programming problem

Maximize $Z = 5X_1 - 2X_2 + 3X_3$

Subject to

$$2X_1 + 2X_2 - X_3 \geq 2$$

$$3X_1 - 4X_2 \leq 3$$

$$X_2 + 3X_3 \leq 5$$

$$X_1, X_2, X_3 \geq 0$$

Linear Programming (LP)

Dual Problem – Example 1

Primal

Maximize $Z = 5X_1 - 2X_2 + 3X_3$

Subject to

$$2X_1 + 2X_2 - X_3 \geq 2 \quad y_1$$

$$3X_1 - 4X_2 + 0X_3 \leq 3 \quad y_2$$

$$0X_1 + X_2 + 3X_3 \leq 5 \quad y_3$$

$$X_1, X_2, X_3 \geq 0$$

Dual

Minimize $W = 2y_1 + 3y_2 + 5y_3$

Subject to

$$2y_1 + 3y_2 + 0y_3 \geq 5$$

$$2y_1 - 4y_2 + y_3 \geq -2$$

$$-y_1 + 0y_2 + 3y_3 \geq 3$$

$$y_1 \leq 0, y_2 \geq 0, y_3 \geq 0$$

Linear Programming (LP)

Dual Problem – Example 1

Taking $y_1 = -y_1'$ in order to bring the problem in to the form that can be solved by simplex method, we get

$$\text{Minimize } W = 2y_1 + 3y_2 + 5y_3$$

Subject to

$$2y_1 + 3y_2 + 0y_3 \geq 5$$

$$2y_1 - 4y_2 + y_3 \geq -2$$

$$-y_1 + 0y_2 + 3y_3 \geq 3$$

$$y_1 \leq 0, y_2 \geq 0, y_3 \geq 0$$



$$\text{Minimize } W = -2y_1' + 3y_2 + 5y_3$$

Subject to

$$-2y_1' + 3y_2 + 0y_3 \geq 5$$

$$-2y_1' - 4y_2 + y_3 \geq -2$$

$$y_1' + 0y_2 + 3y_3 \geq 3$$

$$y_1', y_2, y_3 \geq 0$$

Now can be solved by using simplex method

Linear Programming (LP)

Dual Problem – Example 2

Write the dual of the following linear programming problem

Maximize $Z = 5X_1 + 8X_2 + 10X_3$

Subject to

$$X_1 + X_2 + 2X_3 \leq 120$$

$$3X_1 - 2X_2 - X_3 \geq 90$$

$$2X_1 + 4X_2 + 2X_3 = 100$$

$$X_1, X_2, X_3 \geq 0$$

Linear Programming (LP)

Dual Problem – Example 2

Maximize $Z = 5X_1 + 8X_2 + 10X_3$

Subject to

$$X_1 + X_2 + 2X_3 \leq 120$$

$$3X_1 - 2X_2 - X_3 \geq 90$$

$$2X_1 + 4X_2 + 2X_3 = 100$$

$$X_1, X_2, X_3 \geq 0$$

Minimize $W = 120y_1 + 90y_2 + 100y_3$

Subject to

$$y_1 + 3y_2 + 2y_3 \geq 5$$

$$y_1 - 2y_2 + 4y_3 \geq 8$$

$$2y_1 - y_2 + 2y_3 \geq 10$$

$$y_1 \geq 0, y_2 \leq 0, y_3 \text{ unrestricted}$$

Linear Programming (LP)

Dual Problem – Example 2

Lets try to bring it to the form, that can be solved through simplex method, by taking $y_2 = -y_2'$ and $y_3 = y_3' - y_3''$

Minimize $W = 120y_1 - 90y_2' + 100(y_3' - y_3'')$

Subject to

$$y_1 - 3y_2' + 2(y_3' - y_3'') \geq 5$$

$$y_1 + 2y_2' + 4(y_3' - y_3'') \geq 8$$

$$2y_1 + y_2' + 2(y_3' - y_3'') \geq 10$$

$$y_1, y_2', y_3', y_3'' \geq 0$$

Suggested Reading List

1. S D Sharma, Operations Research – Theory, Methods and Applications, 15th edition, Kedar Nath Ram Nath
2. H A Taha, Operations Research – An Introduction, 8th edition, Pearson Prentice Hall
3. F S Hillier, G J Liberman, Introduction to Operations Research, 9th edition, McGrawHill publication
4. G. Srinivasan, Operations research – principles and applications, 2nd edition, PHI Learning

OPERATIONS RESEARCH

UNIT 2

Lecture 1

~~Presented By:~~

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Lecture Details:

Transportation Problem – Introduction & Formulation
Solution – Balanced Transportation Problem

LEARNING OUTCOMES

Upon completion of this lecture, student should be able to

- Formulate given logistic problem into transportation problem
- Apply different techniques to get initial basic feasible solution
- Solve given balanced transportation problem for optimal solution

Transportation Problem - Introduction

- Transportation problem is a linear programming problem.
- In general this problem is concerned with distributing any commodity from any group of supply centers, called **sources**, to any group of receiving centers, called **destinations**, in such a way to minimize the total distribution/transportation cost.
- This problem involves in determining how to transport goods optimally i.e., transport items at minimum cost from the sources to the destinations.
- This problem is also called as Hitchcock problem based on its origin from Hitchcock.

Transportation Problem - Formulation

Statement of the problem

- A single commodity has to be transported from m sources to n destinations
- The supply capacity of source i is a_i and demand in destination j is b_j .
- The unit cost of transportation from source i to destination j is C_{ij} .
- The problem is to transport the item at minimum cost from the different sources to the different destinations.

Transportation Problem - Formulation

	1	2	3	4	Supply
1	C_{11}	C_{12}	C_{13}	C_{14}	S_1
2	C_{21}	C_{22}	C_{23}	C_{24}	S_2
3	C_{31}	C_{32}	C_{33}	C_{34}	S_3
Demand	D_1	D_1	D_3	D_4	

Transportation Problem - Formulation

$$\text{Minimize} \quad \sum_{i=1}^m \sum_{j=1}^n C_{ij} X_{ij}$$

subjected to

$$\sum_{j=1}^n X_{ij} \leq a_i \quad \forall i = 1, 2, \dots, m$$

$$\sum_{i=1}^m X_{ij} \leq b_j \quad \forall j = 1, 2, \dots, n$$

$$X_{ij} \geq 0 \quad \forall i \text{ and } j$$

Transportation Problem - Formulation

- The constraints ensure that the quantity of item transported from every source is less than or equal to the quantity available and quantity of item that reaches every destination is greater than or equal to the demand
- The decision values are continuous and are not necessarily integer valued.
- In order for the demand at all destination points to be met, it is necessary that $\sum a_i \geq \sum b_j$.

Transportation Problem - Formulation

- If we have $\sum a_i = \sum b_j$, every source will send exactly all that is available to the various destinations and each destination will get exactly what is required. These kind of transportation problems, where the total supply is equal to total demand are called as balanced transportation problems.
- In balanced transportation problems, all constraints are equations.
- Transportation problems in which the total supply is not equal to the total demand are called as unbalanced transportation problems.

Transportation Problem - Formulation

$$\text{Minimize } \sum_{i=1}^m \sum_{j=1}^n C_{ij} X_{ij}$$

subjected to

$$\sum_{j=1}^n X_{ij} = a_i \quad \forall i = 1, 2, \dots, m$$

$$\sum_{i=1}^m X_{ij} = b_j \quad \forall j = 1, 2, \dots, n$$

$$X_{ij} \geq 0 \quad \forall i \text{ and } j$$

Solution Methods – Initial Basic Feasible Solution

- Balanced transportation problems are solved by creating an initial basic feasible solution, that is close to optimum and then reaching the optimum in very few iterations.
- The methods used to create basic feasible solutions to balanced transportation problems are
 1. Northwest corner rule
 2. Least cost method
 3. Vogel's approximation method

Solution Methods – Initial Basic Feasible Solution

- A general transportation problem with m sources and n destinations has $m + n$ constraint equations.
- However, because the transportation model is always balanced, one of the equations is redundant.
- Thus the model has $m + n - 1$ independent constraint equations, which means that the starting basic solution consists of $m + n - 1$ basic variables.
- In general, not always, the Vogel method yields best starting basic solution, and the Northwest corner method yields the worst. The tradeoff is that the Northwest corner method involves the least amount of computations.

Northwest Corner Method (starts at northwest corner cell)

- Allocate as much as possible to the selected cell, and adjust the associated amounts of supply and demand by subtracting the allocated amount.
- Cross out the row or column with zero supply or demand to indicate that no further assignments can be made in that row or column. If both a row and a column net to zero simultaneously, cross out one only, and leave a zero supply (demand) in the uncrossed – out row (column) .
- If exactly one row or column is left uncrossed out, stop. Otherwise, move to the cell to the right if a column has just been crossed out or below if a row has been crossed out. Go to step 1.

Northwest Corner Method - Example

A logistic company VDL ships loads of pickles from three factories of company RIYA to four outlets, the supply (in loads) and the demand (also in loads) together with the unit transportation costs per load on the different routes are given in the following table. (Unit cost – thousands of rupees)

	1	2	3	4	Supply
1	4	6	8	8	40
2	6	8	6	7	60
3	5	7	6	8	50
Demand	20	30	50	50	

Northwest Corner Method - Example

	1	2	3	4	Supply
1	4	6	8	8	40
2	6	8	6	7	60
3	5	7	6	8	50
Demand	20	30	50	50	

Northwest Corner Method - Example

	1	2	3	4	Supply
1	20				20
2					60
3					50
Demand	0	30	50	50	

Northwest Corner Method - Example

	1	2	3	4	Supply
1	20	4	6	8	20
2		6	8	6	60
3		5	7	6	50
Demand	0	30	50	50	

Northwest Corner Method - Example

	1	2	3	4	Supply
1	20	20			0
2					60
3					50
Demand	0	10	50	50	

Northwest Corner Method - Example

	1	2	3	4	Supply
1	4	6	8	8	0
2	20	20			60
3	6	8	6	7	50
Demand	0	10	50	50	

Northwest Corner Method - Example

	1	2	3	4	Supply
1	4	6	8	8	0
2	20	20			50
3	6	8	6	7	50
Demand	0	0	50	50	

Northwest Corner Method - Example

	1	2	3	4	Supply
1	4	6	8	8	0
2	20	20			50
3	6	8	6	7	50
Demand	0	0	50	50	

Northwest Corner Method - Example

	1	2	3	4	Supply
1	4	6	8	8	0
2	6	8	6	7	0
3	5	7	6	8	50
Demand	0	0	0	50	

Northwest Corner Method - Example

	1	2	3	4	Supply
1	4	6	8	8	0
2	6	8	6	7	0
3	5	7	6	8	50
Demand	0	0	0	50	

Northwest Corner Method - Example

	1	2	3	4	Supply
1	4	6	8	8	0
2	6	8	6	7	0
3	5	7	6	8	0
Demand	0	0	0	0	

North West Corner Method - Example

- The initial basic feasible solution is

	1	2	3	4	Supply
1	20	20			0
2		10	50		0
3				50	0
Demand	0	0	0	0	

- The associated cost of the schedule is

$$Z = 20 \times 4 + 20 \times 6 + 10 \times 8 + 50 \times 6 + 50 \times 8 = 980$$

Least Cost Method

1. Assign as much as possible to the cell with the smallest unit cost (ties are broken arbitrarily)
2. The satisfied row or column is crossed out and the amounts of supply and demand are adjusted accordingly.
3. If both a row and a column are satisfied simultaneously, only one is crossed out, the same is in the north west corner method.
4. Look for the uncrossed-out cell with the smallest unit cost and repeat the process until exactly one row or column is left uncrossed-out

Least Cost Method - Example

	1	2	3	4	Supply
1	4	6	8	8	40
2	6	8	6	7	60
3	5	7	6	8	50
Demand	20	30	50	50	

Least Cost Method - Example

	1	2	3	4	Supply
1	20				20
2					60
3					50
Demand	0	30	50	50	

Least Cost Method - Example

	1	2	3	4	Supply
1	20	4	6	8	20
2		6	8	6	60
3		5	7	6	50
Demand	0	30	50	50	

Least Cost Method - Example

	1	2	3	4	Supply
1	20	20			0
2					60
3					50
Demand	0	10	50	50	

Least Cost Method - Example

	1		2	3	4	Supply
		4	6	8	8	
1	20		20			0
2		6	8	6	7	60
3		5	7	6	8	50
Demand	0		10	50	50	

Least Cost Method - Example

	1		2	3	4	Supply
		4	6	8	8	
1	20		20			0
2		6	8	6	7	10
3		5	7	6	8	50
Demand	0		10	0	50	

Least Cost Method - Example

	1	2	3	4	Supply
1	20	20	0	0	0
2			50		10
3					50
Demand	0	10	0	50	

Least Cost Method - Example

	1	2	3	4	Supply
1	20	20	50	10	0
2					0
3					50
Demand	0	10	0	40	

Least Cost Method - Example

	1	2	3	4	Supply
1	20	20			0
2			50	10	0
3					50
Demand	0	10	0	40	

Least Cost Method - Example

	1	2	3	4	Supply
1	20	20			0
2			50	10	0
3		10			40
Demand	0	0	0	40	

Least Cost Method - Example

	1	2	3	4	Supply
1	20	20			0
2			50	10	0
3		10			40
Demand	0	0	0	40	

Least Cost Method - Example

	1	2	3	4	Supply
1	20	20			0
2			50	10	0
3		10		40	0
Demand	0	0	0	0	

Least Cost Method - Example

- The initial basic feasible solution is

	1	2	3	4	Supply
1	20	20			0
2			50	10	0
3		10		40	0
Demand	0	0	0	0	

- The associated cost of the schedule is

$$Z = 20 \times 4 + 20 \times 6 + 50 \times 6 + 10 \times 7 + 10 \times 7 + 40 \times 8 = 960$$

Vogel's Approximation Method

- For each row (column), determine a penalty measure by subtracting the smallest unit cost element in the row (column) from the next smallest unit cost element in the same row (column).
- Identify the row or column with the largest penalty. Break ties arbitrarily. Allocate as much as possible to the variable with the least unit cost in the selected row or column. Adjust the supply and demand, and cross out the satisfied row or column. If a row and a column are satisfied simultaneously, only one of the two is crossed out, and the remaining row (column) is assigned zero supply (demand).

Vogel's Approximation Method

Next....

- ✓ If exactly one row or column with zero supply or demand remains uncrossed-out, stop.
- ✓ If one row (column) with positive supply (demand) remains uncrossed out, determine the basic variables in the row (column) by the least cost method. Stop.
- ✓ If all the uncrossed out rows and columns have (remaining) zero supply and demand, determine the zero basic variables by the least cost method. Stop.

Vogel's Approximation Method - Example

	1	2	3	4	Supply	Row penalty
1	4	6	8	8	40	
2	6	8	6	7	60	
3	5	7	6	8	50	
Demand	20	30	50	50		
Column penalty						

Vogel's Approximation Method - Example

	1	2	3	4	Supply	Row penalty
1	4	6	8	8	40	2
2	6	8	6	7	60	0
3	5	7	6	8	50	1
Demand	20	30	50	50		
Column penalty	1	1	0	1		

Vogel's Approximation Method - Example

	1	2	3	4	Supply	Row penalty
1	20	4	6	8	20	2
2		6	8	6	60	0
3		5	7	6	50	1
Demand	0	30	50	50		
Column penalty	1	1	0	1		

Vogel's Approximation Method - Example

	1	2	3	4	Supply	Row penalty
1	20	4	6	8	20	2
2		6	8	6	60	1
3		5	7	6	50	1
Demand	0	30	50	50		
Column penalty	1	1	0	1		

Vogel's Approximation Method - Example

	1	2	3	4	Supply	Row penalty
1	20	20			0	2
2					60	1
3					50	1
Demand	0	10	50	50		
Column penalty	1	1	0	1		

Vogel's Approximation Method - Example

		1	2	3	4	Supply	Row penalty
		4	6	8	8		
1	20		20			0	2
2		6	8	6	7	60	1
3		5	7	6	8	50	1
Demand		0	10	50	50		
Column penalty		1	1	0	1		

Vogel's Approximation Method - Example

		1	2	3	4	Supply	Row penalty
		4	6	8	8		
1	20		20			0	2
2		6	8	6	7	60	1
3		5	7	6	8	0	1
Demand		0	10	0	50		
Column penalty		1	1	0	1		

Vogel's Approximation Method - Example

	1	2	3	4	Supply	Row penalty
1	20	20	50	0	0	2
2					60	1
3					0	1
Demand	0	10	0	50		
Column penalty	1	1	0	1		

Vogel's Approximation Method - Example

	1	2	3	4	Supply	Row penalty
1	20	20			0	2
2				50	10	1
3			50		0	1
Demand	0	10	0	0		
Column penalty	1	1	0	1		

Vogel's Approximation Method - Example

	1	2	3	4	Supply	Row penalty
1	20	20			0	2
2				50	10	-
3			50		0	-
Demand	0	10	0	0		
Column penalty	1	1	0	1		

Vogel's Approximation Method - Example

	1	2	3	4	Supply	Row penalty
1	20	20			0	2
2		10		50	0	-
3			50		0	-
Demand	0	0	0	0		
Column penalty	1	-	0	1		

Vogel's Approximation Method - Example

- The initial basic feasible solution is

	1	2	3	4	Supply
1	20	20			0
2		10		50	0
3			50		0
Demand	0	0	0	0	

- The associated scheduling cost is

$$Z = 20 \times 4 + 20 \times 6 + 10 \times 8 + 50 \times 7 + 50 \times 6 = 930$$

Optimal Solution

- Number of allocations made in starting basic feasible solutions should be equal to $m + n - 1$. If there are less allotments than this, then the initial solution obtained is degenerate.
- In the present example, the number of allotments (basic variables) should be $3 + 4 - 1 = 6$.
- Vogel's method results in 5 allotments and hence gives a degenerate solution.
- Northwest corner, least cost methods allocates 6 cells and hence are not degenerate.
- Before proceeding to optimal iterations, one should check for degeneracy in initial basic feasible solution.

Optimal Solution

Modified Distribution (MODI) Method

Iteration 1

- Start with initial basic feasible solution obtained from either of the three methods discussed earlier.

	1	2	3	4	Supply
1	4 20	6 20	8	8	0
2	6	8	6 50	7 10	0
3	5	7 10	6	8 40	0
Demand	0	0	0	0	

Optimal Solution Modified Distribution (MODI) Method

- Multipliers u_i and v_j are associated with each i^{th} row and j^{th} column of the transportation table

	v_1	v_2	v_3	v_4	
	1	2	3	4	Supply
u_1	1	2	3	4	0
	4	6	8	8	
	20	20			
u_2	2				0
	6	8	6	7	
			50	10	
u_3	3				0
	5	7	6	8	
		10		40	
Demand	0	0	0	0	

Optimal Solution Modified Distribution (MODI) Method

- Solve for all u_i and v_j values by setting $u_1 = 0$ and using $u_i + v_j = C_{ij}$ for all allotted cells.

		$v_1 = 4$	$v_2 = 6$	$v_3 = 6$	$v_4 = 7$	
		1	2	3	4	Supply
$u_1 = 0$	1	4 20	6 20	8	8	0
$u_2 = 0$	2	6	8	6 50	7 10	0
$u_3 = 1$	3	5	7 10	6	8 40	0
	Demand	0	0	0	0	

Optimal Solution

Modified Distribution (MODI) Method

- Calculate opportunity cost of each unallocated cell using the formula $C_{ij} - u_i - v_j$

		$v_1 = 4$		$v_2 = 6$		$v_3 = 6$		$v_4 = 7$		
		1		2		3		4		S
$u_1 = 0$	1	20	4	20	6		8		8	40
							2		1	
$u_2 = 0$	2		6		8	50	6	10	7	60
			2		2					
$u_3 = 1$	3		5	10	7		6	40	8	50
			0				-1			
	D	20		30		50		50		

Optimal Solution

Modified Distribution (MODI) Method

- If the opportunity cost of all unallocated cells is zero or positive, the solution is optimal, and stop.
- otherwise further iterations are required to obtain optimal solution. In that case proceed as follows.
- Select unallocated cell with largest negative opportunity cost as the entering variable (so one of the allocated cell should leave the solution).
- Construct a closed loop that starts and ends at the entering variable cell. The loop consists of connected horizontal and vertical segments only (no diagonal lines).
- Except for the entering variable cell, each corner of the closed loop must coincide with allocated cell

Optimal Solution Modified Distribution (MODI) Method

		$v_1 = 4$		$v_2 = 6$		$v_3 = 6$		$v_4 = 7$		
		1		2		3		4		S
$u_1 = 0$	1	20	4	20	6		8		8	40
							2		1	
$u_2 = 0$	2		6		8	50	6	10	7	60
			2		2					
$u_3 = 1$	3		5	10	7		6	40	8	50
			0				-1			
	D	20		30		50		50		

Optimal Solution Modified Distribution (MODI) Method

		$v_1 = 4$		$v_2 = 6$		$v_3 = 6$		$v_4 = 7$		
		1		2		3		4		S
$u_1 = 0$	1	20	4	20	6		8		8	40
							2		1	
$u_2 = 0$	2		6		8		6		7	60
			2		2					
$u_3 = 1$	3		5	10	7		6		8	50
			0				-1		40	
	D	20		30		50		50		

Optimal Solution

Modified Distribution (MODI) Method

- Assign an amount θ to the entering variable cell. For the supply and demand limits to remain satisfied, alternate between subtracting and adding the amount θ at the successive corners of the loop.
- Determine θ value by considering the following conditions
 - ✓ Supply and demand requirements remain satisfied.
 - ✓ Shipments through all routes remain nonnegative.
 - ✓ Add as maximum value as possible for θ without violating above two conditions

Optimal Solution Modified Distribution (MODI) Method

		$v_1 = 4$		$v_2 = 6$		$v_3 = 6$		$v_4 = 7$		
		1		2		3		4		S
$u_1 = 0$	1	20	4	20	6		8		8	40
							2		1	
$u_2 = 0$	2		6		8		6		7	60
			2		2					
$u_3 = 1$	3		5	10	7		6		8	50
			0				-1			
	D	20		30		50		50		

Diagram illustrating the MODI method for finding the optimal solution. The table shows the current solution with dual values u_i and v_j calculated. The cell (3, 3) is highlighted in green, indicating the entering variable. The path for the leaving variable is shown with dashed lines and θ values.

Optimal Solution Modified Distribution (MODI) Method

		$v_1 = 4$		$v_2 = 6$		$v_3 = 6$		$v_4 = 7$		
		1		2		3		4		S
$u_1 = 0$	1	20	4	20	6		8		8	40
							2		1	
$u_2 = 0$	2		6		8		6		7	60
			2		2					
$u_3 = 1$	3		5	10	7	40	6		8	50
			0				-1		0	
	D	20		30		50		50		

Optimal Solution Modified Distribution (MODI) Method

- Iteration 2

		v_1		v_2		v_3		v_4		
		1		2		3		4		S
u_1	1	20	4	20	6		8		8	40
u_2	2		6		8	10	6	50	7	60
u_3	3		5	10	7	40	6		8	50
D		20		30		50		50		

Optimal Solution Modified Distribution (MODI) Method

- Iteration 2

		$v_1 = 4$		$v_2 = 6$		$v_3 = 5$		$v_4 = 6$		
		1		2		3		4		S
$u_1 = 0$	1	20	4	20	6		8		8	40
$u_2 = 1$	2		6		8	10	6	50	7	60
$u_3 = 1$	3		5	10	7	40	6		8	50
	D	20		30		50		50		

Optimal Solution

Modified Distribution (MODI) Method

- Iteration 2

		$v_1 = 4$		$v_2 = 6$		$v_3 = 5$		$v_4 = 6$		
		1		2		3		4		S
$u_1 = 0$	1	20	4	20	6		8		8	40
							3		2	
$u_2 = 1$	2		6		8	10	6	50	7	60
			1		1					
$u_3 = 1$	3		5	10	7	40	6		8	50
			0						1	
	D	20		30		50		50		

- Since all opportunity costs are ≥ 0 , the solution is optimal.

Optimal Solution Modified Distribution (MODI) Method

- The optimal solution is

	1		2		3		4		S
1	20	4	20	6		8		8	40
2		6		8	10	6	50	7	60
3		5	10	7	40	6		8	50
D	20		30		50		50		

- The associated optimal cost is

$$Z = 20 \times 4 + 20 \times 6 + 10 \times 6 + 50 \times 7 + 10 \times 7 + 40 \times 6 = 920$$

Suggested Reading List

1. F S Hillier, G J Liberman, Introduction to Operations Research, 9th edition, McGrawHill publication
2. G. Srinivasan, Operations research – principles and applications, 2nd edition, PHI Learning
3. H A Taha, Operations Research – An Introduction, 8th edition, Pearson Prentice Hall
4. S D Sharma, Operations Research – Theory, Methods and Applications, 5th edition, Kedarnath - Ramnath & Co.

For more course related information, please visit

<https://sites.google.com/view/operationsresearchgiet/home>

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OPERATIONS RESEARCH

UNIT 2

Lecture 2

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Lecture Details:

Transportation Problem – Degeneracy
Solution – Unbalanced Transportation Problem

LEARNING OUTCOMES

Upon completion of this lecture, student should be able to

- Illustrate the degeneracy in the transportation problems
- Obtain optimal solution to unbalanced transportation problems

Transportation Problem - Degeneracy

- Solve the given problem, use Vogel's approximation method for initial basic feasible solution.

	1	2	3	4	S
1	4	6	8	8	40
2	6	8	6	7	60
3	5	7	6	8	50
D	20	30	50	50	

Transportation Problem - Degeneracy

- Considering initial basic feasible solution by Vogel's approximation method, the total number of allocated cells are 5, which is less than $m + n - 1 = 3 + 4 - 1 = 6$.
- That is one of the basic variable is zero, or the solution is degenerate

	1	2	3	4	S
1	4 20	6 20	8	8	40
2	6	8 10	6	7 50	60
3	5	7	6 50	8	50
D	20	30	50	50	

Transportation Problem - Degeneracy

- Now applying MODI method to obtain the optimal solution,

	v_1	v_2	v_3	v_4	
	1	2	3	4	Supply
u_1	1	2	3	4	40
	4	6	8	8	
	20	20			
u_2	2				60
	6	8	6	7	
		10		50	
u_3	3				50
	5	7	6	8	
			50		
Demand	20	30	50	50	

Transportation Problem - Degeneracy

- Now applying MODI method to obtain the optimal solution,

		$v_1 = 4$	$v_2 = 6$	$v_3 = ?$	$v_4 = 5$	
		1	2	3	4	Supply
$u_1 = 0$	1	4 20	6 20	8	8	40
$u_2 = 2$	2	6	8 10	6	7 50	60
$u_3 = ?$	3	5	7	6 50	8	50
	Demand	20	30	50	50	

Can't find values of u_3 and v_3

Transportation Problem - Degeneracy

- Sixth allocation could be any of X_{13} , X_{14} , X_{21} , X_{23} , X_{31} , X_{32} , $X_{34} = 0$. This is represented by ϵ , X_{13} in the solution

		$v_1 = 4$	$v_2 = 6$	$v_3 = 8$	$v_4 = 5$	
		1	2	3	4	Supply
$u_1 = 0$	1	4 20	6 20	8 ϵ	8	40
$u_2 = 2$	2	6	8 10	6	7 50	60
$u_3 = -2$	3	5	7	6 50	8	50
	Demand	20	30	50	50	

Transportation Problem - Degeneracy

- Now calculating the opportunity costs of all unallocated cells, we get

		$v_1 = 4$		$v_2 = 6$		$v_3 = 8$		$v_4 = 5$		
		1		2		3		4		Supply
$u_1 = 0$	1	20	4	20	6	ϵ	8		8	40
									3	
$u_2 = 2$	2		6	10	8		6	50	7	60
			0				-4			
$u_3 = -2$	3		5		7	50	6		8	50
			3		3				5	
	Demand		20		30		50		50	

Transportation Problem - Degeneracy

- Construct the loop by selecting cell with largest negative opportunity cost as entering variable.

		$v_1 = 4$		$v_2 = 6$		$v_3 = 8$		$v_4 = 5$		
		1		2		3		4		Supply
$u_1 = 0$	1		4		6		8		8	40
		20		20		€			3	
$u_2 = 2$	2		6		8		6		7	60
			0	10			-4	50		
$u_3 = -2$	3		5		7		6		8	50
			3		3	50			5	
	Demand	20		30		50		50		

Transportation Problem - Degeneracy

- Allocate θ to entering variable cell, considering non negativity restriction and demand, supply constraints, we get

		$v_1 = 4$		$v_2 = 6$		$v_3 = 8$		$v_4 = 5$	
		1		2		3		4	S
$u_1 = 0$	1	20	4	$20 + \theta$	6	$\epsilon + \theta$	8	8	40
								3	
$u_2 = 2$	2		6	$10 - \theta$	8	θ	6	7	60
			0				-4	50	
$u_3 = -2$	3		5		7		6	8	50
			3		3	50		5	
	D	20		30		50		50	

Transportation Problem - Degeneracy

		v_1		v_2		v_3		v_4	
		1		2		3		4	S
u_1	1	20	4	20	6		8	8	40
u_2	2		6	10	8	€	6	50	60
u_3	3		5		7	50	6	8	50
	D	20		30		50		50	

Transportation Problem - Degeneracy

		$v_1 = 4$		$v_2 = 6$		$v_3 = 4$		$v_4 = 5$		
		1		2		3		4		S
$u_1 = 0$	1	20	4	20	6		8		8	40
							4		3	
$u_2 = 2$	2		6	10	8	€	6	50	7	60
			0							
$u_3 = 2$	3		5		7	50	6		8	50
			-1		-1				1	
	D	20		30		50		50		

Transportation Problem - Degeneracy

		$v_1 = 4$		$v_2 = 6$		$v_3 = 4$		$v_4 = 5$		
		1		2		3		4		S
$u_1 = 0$	1	20	4	20	6		8		8	40
							4		3	
$u_2 = 2$	2		6	10	8		6	50	7	60
			0							
$u_3 = 2$	3		5		7	50	6		8	50
			-1		-1				1	
	D	20		30		50		50		

Transportation Problem - Degeneracy

		$v_1 = 4$		$v_2 = 6$		$v_3 = 4$		$v_4 = 5$		
		1		2		3		4		S
$u_1 = 0$	1	20	4	20	6		8		8	40
			$-\theta$		$+\theta$		4		3	
$u_2 = 2$	2		6	10	8		6	50	7	60
			0		$-\theta$		$\epsilon + \theta$			
$u_3 = 2$	3		5		7		6		8	50
			θ		-1		$50 - \theta$		1	
	D	20		30		50		50		

Transportation Problem - Degeneracy

		$v_1 = 4$		$v_2 = 6$		$v_3 = 4$		$v_4 = 5$		
		1		2		3		4		S
$u_1 = 0$	1		4		6		8		8	40
		10		30			4		3	
$u_2 = 2$	2		6		8		6		7	60
			0	0		10		50		
$u_3 = 2$	3		5		7		6		8	50
		10	-1		-1	40			1	
	D	20		30		50		50		

Transportation Problem - Degeneracy

	v_1	v_2	v_3	v_4	
	1	2	3	4	S
u_1	1	2	3	4	40
	10	30		8	
u_2	2		10	50	60
				7	
u_3	3		40		50
	10			8	
D	20	30	50	50	

Transportation Problem - Degeneracy

		$v_1 = 4$		$v_2 = 6$		$v_3 = 5$		$v_4 = 6$		
		1		2		3		4		S
$u_1 = 0$	1	10	4	30	6		8		8	40
$u_2 = 1$	2		6		8	10	6	50	7	60
$u_3 = 1$	3	10	5		7	40	6		8	50
	D	20		30		50		50		

Transportation Problem - Degeneracy

		$v_1 = 4$		$v_2 = 6$		$v_3 = 5$		$v_4 = 6$		
		1		2		3		4		S
$u_1 = 0$	1	10	4	30	6		8		8	40
							3		2	
$u_2 = 1$	2		6		8	10	6	50	7	60
			1		1					
$u_3 = 1$	3	10	5		7	40	6		8	50
					0				1	
	D	20		30		50		50		

- Since the opportunity cost of all unallocated cells is ≥ 0 , the solution is optimal.

Transportation Problem - Degeneracy

- The optimal solution is

	1		2		3		4		S
1	10	4	30	6		8		8	40
2		6		8		6		7	
3	10	5		7	40	6		8	50
D									
	20		30		50		50		

- The associated optimal cost is,

$$Z = 10 \times 4 + 30 \times 6 + 10 \times 6 + 50 \times 7 + 10 \times 5 + 40 \times 6 = 920$$

Unbalanced Transportation Problem

A product is produced by three factories I, II and III. Their production capacities are 10, 80 and 15 units respectively. These factories supply the product to three stores, demands of which are 75, 20 and 50 units respectively. Unit transportation cost in rupees from each factory to each store is given in the table below.

Determine the extent of deliveries from each of the factories to each of the stores so that the transportation cost is minimum.

Unbalanced Transportation Problem

	1	2	3	Sup
I	5	1	7	10
II	6	4	6	80
III	3	2	5	15
Dem	75	20	50	

Unbalanced Transportation Problem

	1	2	3	Sup
I	5	1	7	10
II	6	4	6	80
III	3	2	5	15
Dem	75	20	50	

Total Demand = $75+20+50 = 145$

Total Supply = $10+80+15 = 105$

Not Equal

UNBALANCED

Unbalanced Transportation Problem

- Transportation algorithm is based on the assumption that the model is balanced. If the model is unbalanced, a dummy source or a dummy destination is added to restore the balance.
- The cost entries associated with the dummy destination/source should be zero because there is no cost incurred by a fictional allocation.
- Number of units, if allocated from dummy source to destination, presents in optimal solution, then it equals the shortage felt by that destination to satisfy its demand.
- Number of units, if allocated from a source to a dummy destination, presents in optimal solution, then it equals the excess of units remain unshipped with that source.

Unbalanced Transportation Problem

- Since there is an excess demand, a dummy source is used to balance the given problem.

	1	2	3	Sup
I	5	1	7	10
II	6	4	6	80
III	3	2	5	15
IV(D)	0	0	0	40
Dem	75	20	50	

Unbalanced Transportation Problem

- Now determine initial basic feasible solution using Northwest corner method

	1	2	3	Sup
I	5 10	1	7	0
II	6	4	6	80
III	3	2	5	15
IV(D)	0	0	0	40
Dem	65	20	50	

Unbalanced Transportation Problem

- Now determine initial basic feasible solution using Northwest corner method

	1		2		3		Sup
I		5		1		7	0
	10						
II		6		4		6	80
III		3		2		5	15
IV(D)		0		0		0	40
Dem	65		20		50		

Unbalanced Transportation Problem

- Now determine initial basic feasible solution using Northwest corner method

	1		2		3		Sup
I		5		1		7	0
	10						
II		6		4		6	15
	65						
III		3		2		5	15
IV(D)		0		0		0	40
Dem	0		20		50		

Unbalanced Transportation Problem

- Now determine initial basic feasible solution using Northwest corner method

		1	2	3	Sup
I		5	1	7	0
	10				
II		6	4	6	15
	65				
III		3	2	5	15
IV(D)		0	0	0	40
Dem		0	20	50	

Unbalanced Transportation Problem

- Now determine initial basic feasible solution using Northwest corner method

		1	2	3	Sup
I		5	1	7	0
	10				
II		6	4	6	0
	65		15		
III		3	2	5	15
IV(D)		0	0	0	40
Dem		0	5	50	

Unbalanced Transportation Problem

- Now determine initial basic feasible solution using Northwest corner method

		1	2	3	Sup
I		5	1	7	0
	10				
II		6	4	6	0
	65		15		
III		3	2	5	15
IV(D)		0	0	0	40
Dem		0	5	50	

Unbalanced Transportation Problem

- Now determine initial basic feasible solution using Northwest corner method

		1	2	3	Sup
I		5	1	7	0
	10				
II		6	4	6	0
	65		15		
III		3	2	5	10
			5		
IV(D)		0	0	0	40
Dem		0	0	50	

Unbalanced Transportation Problem

- Now determine initial basic feasible solution using Northwest corner method

		1	2	3	Sup
I		5	1	7	0
		10			
II		6	4	6	0
		65	15		
III		3	2	5	10
			5		
IV(D)		0	0	0	40
Dem		0	0	50	

Unbalanced Transportation Problem

- Now determine initial basic feasible solution using Northwest corner method

		1	2	3	Sup
I		5	1	7	0
		10			
II		6	4	6	0
		65	15		
III		3	2	5	0
			5	10	
IV(D)		0	0	0	40
Dem		0	0	40	

Unbalanced Transportation Problem

- Now determine initial basic feasible solution using Northwest corner method

		1	2	3	Sup
I		5	1	7	0
		10			
II		6	4	6	0
		65	15		
III		3	2	5	0
			5	10	
IV(D)		0	0	0	40
Dem		0	0	40	

Unbalanced Transportation Problem

- Now determine initial basic feasible solution using Northwest corner method

		1	2	3	Sup
I		5	1	7	0
		10			
II		6	4	6	0
		65	15		
III		3	2	5	0
			5	10	
IV(D)		0	0	0	0
				40	
Dem		0	0	0	

Unbalanced Transportation Problem

- The initial basic feasible solution is

	1	2	3	Sup
I	5 10	1	7	10
II	6 65	4 15	6	80
III	3	2 5	5 10	15
IV(D)	0	0	0 40	40
Dem	75	20	50	

Unbalanced Transportation Problem

- Now determine optimal solution using MODI method

		v_1	v_2	v_3	
		1	2	3	Sup
u_1	I	5	1	7	10
		10			
u_2	II	6	4	6	80
		65	15		
u_3	III	3	2	5	15
			5	10	
u_4	IV(D)	0	0	0	40
				40	
	Dem	75	20	50	

Unbalanced Transportation Problem

		$v_1 = 5$		$v_2 = 3$		$v_3 = 6$		
		1		2		3		Sup
$u_1 = 0$	I		5		1		7	10
		10						
$u_2 = 1$	II		6		4		6	80
		65		15				
$u_3 = -1$	III		3		2		5	15
				5		10		
$u_4 = -6$	IV(D)		0		0		0	40
						40		
Dem		75		20		50		

Unbalanced Transportation Problem

		$v_1 = 5$		$v_2 = 3$		$v_3 = 6$		
		1		2		3		Sup
$u_1 = 0$	I		5		1		7	10
		10			-2		1	
$u_2 = 1$	II		6		4		6	80
		65		15			-1	
$u_3 = -1$	III		3		2		5	15
			-1	5		10		
$u_4 = -6$	IV(D)		0		0		0	40
			1		3	40		
	Dem	75		20		50		

Unbalanced Transportation Problem

		$v_1 = 5$		$v_2 = 3$		$v_3 = 6$		
		1		2		3		Sup
$u_1 = 0$	I		5		1		7	10
		10			-2		1	
$u_2 = 1$	II		6		4		6	80
		65					-1	
$u_3 = -1$	III		3		2		5	15
			-1	5		10		
$u_4 = -6$	IV(D)		0		0		0	40
			1		3	40		
	Dem	75		20		50		

Unbalanced Transportation Problem

		$v_1 = 5$		$v_2 = 3$		$v_3 = 6$		
		1		2		3		Sup
$u_1 = 0$	I		5		1		7	10
		$10 - \theta$			θ		-2	
$u_2 = 1$	II		6		4		6	80
		$65 + \theta$			$15 - \theta$		-1	
$u_3 = -1$	III		3		2		5	15
			-1	5		10		
$u_4 = -6$	IV(D)		0		0		0	40
			1		3	40		
	Dem	75		20		50		

Unbalanced Transportation Problem

		v_1		v_2		v_3		
		1		2		3		Sup
u_1	I		5		1		7	10
				10				
u_2	II		6		4		6	80
		75		5				
u_3	III		3		2		5	15
				5		10		
u_4	IV(D)		0		0		0	40
						40		
	Dem	75		20		50		

Unbalanced Transportation Problem

		$v_1 = 3$		$v_2 = 1$		$v_3 = 4$		
		1		2		3		Sup
$u_1 = 0$	I		5		1		7	10
				10				
$u_2 = 3$	II		6		4		6	80
		75		5				
$u_3 = 1$	III		3		2		5	15
				5		10		
$u_4 = -4$	IV(D)		0		0		0	40
						40		
	Dem	75		20		50		

Unbalanced Transportation Problem

		$v_1 = 3$		$v_2 = 1$		$v_3 = 4$		
		1		2		3		Sup
$u_1 = 0$	I		5		1		7	10
			2	10			3	
$u_2 = 3$	II		6		4		6	80
		75		5			-1	
$u_3 = 1$	III		3		2		5	15
			-1	5		10		
$u_4 = -4$	IV(D)		0		0		0	40
			1		3	40		
	Dem	75		20		50		

Unbalanced Transportation Problem

		$v_1 = 3$		$v_2 = 1$		$v_3 = 4$		
		1		2		3		Sup
$u_1 = 0$	I		5		1		7	10
			2	10			3	
$u_2 = 3$	II		6		4		6	80
		75		5			-1	
$u_3 = 1$	III		3		2		5	15
			-1	5		10		
$u_4 = -4$	IV(D)		0		0		0	40
			1		3	40		
	Dem	75		20		50		

Unbalanced Transportation Problem

		$v_1 = 3$		$v_2 = 1$		$v_3 = 4$		
		1		2		3		Sup
$u_1 = 0$	I		5		1		7	10
			2	10			3	
$u_2 = 3$	II		6		4		6	80
		$75 - \theta$		$5 + \theta$			-1	
$u_3 = 1$	III		3		2		5	15
		θ	-1	$5 - \theta$		10		
$u_4 = -4$	IV(D)		0		0		0	40
			1		3	40		
	Dem	75		20		50		

Unbalanced Transportation Problem

		v_1		v_2		v_3		
		1		2		3		Sup
u_1	I		5		1		7	10
				10				
u_2	II		6		4		6	80
		70		10				
u_3	III		3		2		5	15
		5				10		
u_4	IV(D)		0		0		0	40
						40		
	Dem	75		20		50		

Unbalanced Transportation Problem

		$v_1 = 3$		$v_2 = 1$		$v_3 = 5$		
		1		2		3		Sup
$u_1 = 0$	I		5		1		7	10
				10				
$u_2 = 3$	II		6		4		6	80
		70		10				
$u_3 = 0$	III		3		2		5	15
		5				10		
$u_4 = -5$	IV(D)		0		0		0	40
						40		
	Dem	75		20		50		

Unbalanced Transportation Problem

		$v_1 = 3$		$v_2 = 1$		$v_3 = 5$		
		1		2		3		Sup
$u_1 = 0$	I		5		1		7	10
			2	10			2	
$u_2 = 3$	II		6		4		6	80
		70		10			-2	
$u_3 = 0$	III		3		2		5	15
		5			1	10		
$u_4 = -5$	IV(D)		0		0		0	40
			2		4	40		
	Dem	75		20		50		

Unbalanced Transportation Problem

		$v_1 = 3$		$v_2 = 1$		$v_3 = 5$		
		1		2		3		Sup
$u_1 = 0$	I		5		1		7	10
			2	10			2	
$u_2 = 3$	II		6		4		6	80
		70		~~10~~			-2	
$u_3 = 0$	III		3		2		5	15
		5			~~1~~	10		
$u_4 = -5$	IV(D)		0		0		0	40
			2		4	40		
	Dem	75		20		50		

Unbalanced Transportation Problem

		$v_1 = 3$		$v_2 = 1$		$v_3 = 5$		
		1		2		3		Sup
$u_1 = 0$	I		5		1		7	10
			2	10			2	
$u_2 = 3$	II		6		4		6	80
		~~$70 - \theta$~~	~~---~~	~~10~~	~~---~~	~~θ~~	-2	
$u_3 = 0$	III		3		2		5	15
		~~$5 + \theta$~~	~~---~~	~~---~~	~~1~~	~~$10 - \theta$~~		
$u_4 = -5$	IV(D)		0		0		0	40
			2		4	40		
	Dem	75		20		50		

Unbalanced Transportation Problem

		v_1		v_2		v_3		
		1		2		3		Sup
u_1	I		5		1		7	10
				10				
u_2	II		6		4		6	80
		60		10		10		
u_3	III		3		2		5	15
		15						
u_4	IV(D)		0		0		0	40
						40		
	Dem	75		20		50		

Unbalanced Transportation Problem

		$v_1 = 3$		$v_2 = 1$		$v_3 = 3$		
		1		2		3		Sup
$u_1 = 0$	I		5		1		7	10
			2	10			4	
$u_2 = 3$	II		6		4		6	80
		60		10		10		
$u_3 = 0$	III		3		2		5	15
		15			1		2	
$u_4 = -3$	IV(D)		0		0		0	40
			0		2	40		
	Dem	75		20		50		

- Since all opportunity costs are ≥ 0 , the solution is optimal

Unbalanced Transportation Problem

- The optimal solution is

	1		2		3		Sup
I		5		1		7	10
			10				
II		6		4		6	80
	60		10		10		
III		3		2		5	15
	15						
IV(D)		0		0		0	40
					40		
Dem	75		20		50		

$$Z_{opt} = 10 \times 1 + 60 \times 6 + 10 \times 4 + 10 \times 6 + 15 \times 3 = 515$$

Suggested Reading List

1. F S Hillier, G J Liberman, Introduction to Operations Research, 9th edition, McGrawHill publication
2. G. Srinivasan, Operations research – principles and applications, 2nd edition, PHI Learning
3. H A Taha, Operations Research – An Introduction, 8th edition, Pearson Prentice Hall
4. S D Sharma, Operations Research – Theory, Methods and Applications, 5th edition, Kedarnath - Ramnath & Co.

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OPERATIONS RESEARCH

UNIT 2

Lecture 3

~~Presented By:~~



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Lecture Details:

Formulation and Solution of Assignment Problem

Operations Research/ME, 5 Semester.

LEARNING OUTCOMES

Upon completion of this lecture, student should be able to

- Formulate mathematical model to assign resources to tasks
- Obtain optimal assignment of resources to tasks using Hungarian Algorithm

Assignment Problem - Introduction

- Assignment problem is a special kind of linear programming problem, which involves assigning resources to tasks to minimize the cost of performing the tasks.
- Number of assignees and tasks are the same
- Resources are may be people, or vehicles, or machines, or plants, or even time slots to be assigned tasks.
- Each task gets only one resource and each resource gets only one task at a time.
- Decision variables in assignment problem takes 1, if resource is allotted to job, 0, otherwise.

Assignment Problem - Formulation

	I	II	III	IV
A	C_{11}	C_{12}	C_{13}	C_{14}
B	C_{21}	C_{22}	C_{23}	C_{24}
C	C_{31}	C_{22}	C_{33}	C_{34}
D	C_{41}	C_{42}	C_{43}	C_{44}

Assignment Problem - Formulation

$$\text{Minimize} \quad \sum_{i=1}^n \sum_{j=1}^n C_{ij} X_{ij}$$

subject to

$$\sum_{i=1}^n X_{ij} = 1 \quad \text{for } j = 1, 2, \dots, n$$

$$\sum_{j=1}^n X_{ij} = 1 \quad \text{for } i = 1, 2, \dots, n$$

$$\text{and} \quad X_{ij} = 0, 1$$

Assignment Problem – Solution Hungarian Algorithm

Step 1

Subtract the smallest number in each row from every number in the row.

Step 2

Subtract the smallest number in each column of the new table from every number in the column.

Step 3

Test whether an optimal set of assignments can be made. This can be done by determining the minimum number of lines needed to cover all zeros. If the minimum number of lines equals the number of rows, go to Step 6 otherwise go to Step 4.

Assignment Problem – Solution Hungarian Algorithm

Step 4

- Subtract the smallest uncovered number from every uncovered number in the table.
- Add the smallest uncovered number to the numbers at intersections of the covering lines.
- Numbers crossed out but not at the intersections of cross out lines carry over unchanged to the next table.

Step 5

Repeat Step 3 and 4 until an optimal set of assignments is possible.

Assignment Problem – Solution

Hungarian Algorithm

Step 6

- Make the assignments one at a time in positions that have zero elements. Begin with rows or columns that have only one zero. Since each row and column needs to receive exactly one assignment, cross out both the row and the column involved after each assignment is made.
- Then move on to the rows and columns that are not yet crossed out to select the next assignment, with preference again given to any such row or column that has only one zero that is not crossed out.
- Continue until every row and every column has exactly one assignment and so has been crossed out.

Assignment Problem – Example 1

Suggest optimum assignment of 4 workers A, B, C and D to 4 jobs I, II, III and IV. The time taken by different workers in completing the different jobs is given below.

	I	II	III	IV
A	8	10	12	16
B	11	11	15	8
C	9	6	5	14
D	15	14	9	7

Also find the total time taken in completing the jobs.

Assignment Problem – Example 1

Row reduction - Subtract the smallest number in each row from every number in the row.

	I	II	III	IV
A	8	10	12	16
B	11	11	15	8
C	9	6	5	14
D	15	14	9	7

Assignment Problem – Example 1

Row reduction - Subtract the smallest number in each row from every number in the row.

	I	II	III	IV
A	0	2	4	8
B	3	3	7	0
C	4	1	0	9
D	8	7	2	0

Assignment Problem – Example 1

Column reduction - Subtract the smallest number in each column of the new table from every number in the column.

	I	II	III	IV
A	0	1	4	8
B	3	2	7	0
C	4	0	0	9
D	8	6	2	0

Assignment Problem – Example 1

Test whether an optimal set of assignments can be made by determining the minimum number of lines.

	I	II	III	IV
A	0	1	4	8
B	3	2	7	0
C	4	0	0	9
D	8	6	2	0

Assignment Problem – Example 1

Test whether an optimal set of assignments can be made by determining the minimum number of lines.

	I	II	III	IV
A	0	1	4	8
B	3	2	7	0
C	4	0	0	9
D	8	6	2	0

Minimum number of lines < Number of rows – **Not Optimal**

Assignment Problem – Example 1

Since the complete assignment is not possible, follow step 4.
smallest uncovered element is 2.

	I	II	III	IV
A	0	1	4	8
B	3	2	7	0
C	4	0	0	9
D	8	6	2	0

Assignment Problem – Example 1

Since the complete assignment is not possible, follow step 4.
smallest uncovered element is 2.

	I	II	III	IV
A	0	1	4	10
B	1	0	5	0
C	4	0	0	11
D	6	4	0	0

Assignment Problem – Example 1

Now repeating step 3 again,

	I	II	III	IV
A	0	1	4	10
B	1	0	5	0
C	4	0	0	11
D	6	4	0	0

- As the minimum number of lines is equal to number of rows, optimal assignment can be made. Go to step 6

Assignment Problem – Example 1

- Make the assignments one at a time in positions that have zero elements.

	I	II	III	IV
A	0	1	4	10
B	1	0	5	0
C	4	0	0	11
D	6	4	0	0

Assignment Problem – Example 1

- Make the assignments one at a time in positions that have zero elements.

	II	III	III	IV
A	0	1	4	10
B	1	0	5	0
C	4	0	0	11
D	6	4	0	0

Assignment Problem – Example 1

- Make the assignments one at a time in positions that have zero elements.

	I	II	III	IV
A	0	1	4	10
B	1	0	5	0
C	4	0	0	11
D	6	4		0

Assignment Problem – Example 1

- Make the assignments one at a time in positions that have zero elements.

	I	II	III	IV
A	0	1	4	10
B	1	0	5	0
C	4	0	0	11
D	6	4		0

Assignment Problem – Example 1

- The optimal assignment is

$$A \rightarrow I, B \rightarrow II, C \rightarrow III, D \rightarrow IV$$

	I	II	III	IV
A	8	10	12	16
B	11	11	15	8
C	9	6	5	14
D	15	14	9	7

- Time taken to complete the jobs

$$8 + 11 + 5 + 7 = 31$$

Assignment Problem – Example 2

Four new machines M_1 , M_2 , M_3 and M_4 are to be installed in a machine shop. There are five vacant places A, B, C, D and E available. Because of limited space, machine M_2 cannot be placed at C and M_3 cannot be placed at A. the assignment cost of machine i to place j in rupees is shown below.

	A	B	C	D	E
M_1	4	6	10	5	6
M_2	7	4	-	5	4
M_3	-	6	9	6	2
M_4	9	3	7	2	3

Find the optimal assignment schedule

Assignment Problem – Example 2

- Since the number of machines and places are not same, a dummy machine is added to make the matrix, square. The costs associated with dummy machine is zero.
- The restricted assignments are given a large cost, so that assignment in those cells cannot be made by the algorithm.

	A	B	C	D	E
M_1	4	6	10	5	6
M_2	7	4	M	5	4
M_3	M	6	9	6	2
M_4	9	3	7	2	3
$M_5(D)$	0	0	0	0	0

Assignment Problem – Example 2

	A	B	C	D	E
M_1	4	6	10	5	6
M_2	7	4	M	5	4
M_3	M	6	9	6	2
M_4	9	3	7	2	3
$M_5(D)$	0	0	0	0	0

- The matrix is now in the form that can be solved by Hungarian algorithm.
- M is very large number (used to restrict assignment)
- Any assignment of dummy machine to a place in optimal solution, can be understood as that place remains vacant.

Suggested Reading List

1. F S Hillier, G J Liberman, Introduction to Operations Research, 9th edition, McGrawHill publication
2. G. Srinivasan, Operations research – principles and applications, 2nd edition, PHI Learning
3. H A Taha, Operations Research – An Introduction, 8th edition, Pearson Prentice Hall
4. P K Gupta, D S Hira, Operations Research, 7th edition, S Chand Publishers

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OPERATIONS RESEARCH

UNIT 2

Lecture 4

~~Presented By:~~



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Lecture Details:

Travelling Salesman Problem

Operations Research/ME, 5 Semester.

LEARNING OUTCOMES

Upon completion of this lecture, student should be able to

- Differentiate feasible and infeasible solution in travelling salesman problem
- Evaluate feasible routes for optimality in travelling salesman problems
- Construct minimum spanning tree for transportation problem

Traveling Salesman Problem - Introduction

- Travelling salesman problem involves finding the shortest distance to be covered by the salesman in order to visit n cities and return to the starting point.
- Since the person comes back to starting point, any of the n cities can be the starting point.
- The person visits the each city once and returns to the starting point.

Traveling Salesman Problem – Feasible Solution

- Satisfying the requirements mentioned for a travelling salesman problem, one can find the feasible solution (may not be the optimal) as shown below (for a general 5 city problem).

	1	2	3	4	5
1	-	d_{12}	d_{13}	d_{14}	d_{15}
2	d_{21}	-	d_{23}	d_{24}	d_{25}
3	d_{31}	d_{32}	-	d_{34}	d_{35}
4	d_{41}	d_{42}	d_{43}	-	d_{45}
5	d_{51}	d_{52}	d_{53}	d_{54}	-

- The feasible solution is

$$1 \rightarrow 3 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 1$$

- The associated cost is

$$Z = \sum_{i=1}^5 \sum_{j=1}^5 X_{ij} d_{ij}$$

- For this particular solution, only X_{13} , X_{24} , X_{32} , X_{45} & $X_{51} = 1$. all other X_{ij} are zero.

Traveling Salesman Problem – Example

A delivery boy has to visit five cities to deliver items ordered online by the customers. He does not want to visit any city twice before completing the tour of all the cities and wishes to return to the city from which he has started. Cost of going from one city to another in rupees is given in the table below. Find the least cost route.

	1	2	3	4	5
1	-	10	8	9	7
2	10	-	10	5	6
3	8	10	-	8	9
4	9	5	8	-	6
5	7	6	9	6	-

Traveling Salesman Problem - Example

Treating the given travelling salesman problem as assignment problem and applying the Hungarian algorithm,

	1	2	3	4	5
1	-	10	8	9	7
2	10	-	10	5	6
3	8	10	-	8	9
4	9	5	8	-	6
5	7	6	9	6	-

Traveling Salesman Problem - Example

- Replacing constraints in the table with large cost M, we get

	1	2	3	4	5
1	M	10	8	9	7
2	10	M	10	5	6
3	8	10	M	8	9
4	9	5	8	M	6
5	7	6	9	6	M

Traveling Salesman Problem - Example

- Row reduction

	1	2	3	4	5
1	M	3	1	2	0
2	5	M	5	0	1
3	0	2	M	0	1
4	4	0	3	M	1
5	1	0	3	0	M

Traveling Salesman Problem - Example

- Column reduction

	1	2	3	4	5
1	M	3	0	2	0
2	5	M	4	0	1
3	0	2	M	0	1
4	4	0	2	M	1
5	1	0	2	0	M

Traveling Salesman Problem - Example

- Check for optimality by determining the minimum number of lines

	1	2	3	4	5
1	M	3	0	2	0
2	5	M	4	0	1
3	0	2	M	0	1
4	4	0	2	M	1
5	1	0	2	0	M

Optimal?

Traveling Salesman Problem - Example

- Creating additional zeros

	1	2	3	4	5
1	M	4	0	3	0
2	4	M	3	0	0
3	0	3	M	1	1
4	3	0	1	M	0
5	0	0	1	0	M

Traveling Salesman Problem - Example

- Repeating check for optimality

	1	2	3	4	5
1	M	4	0	3	0
2	4	M	3	0	0
3	0	3	M	1	1
4	3	0	1	M	0
5	0	0	1	0	M

- Since the minimum number of lines equal the number of rows, the assignment is optimal.

Traveling Salesman Problem - Example

- Making assignments one at a time

	1	2	3	4	5
1	M	4	0	3	0
2	4	M	3	0	0
3	0	3	M	1	1
4	3	0	1	M	0
5	0	0	1	0	M

Traveling Salesman Problem - Example

- Making assignments one at a time

	1	2	3	4	5
1	M	4	0	3	0
2	4	M	3	0	0
3	0	3	M	1	1
4	3	0	1	M	0
5	0	0	1	0	M

Traveling Salesman Problem - Example

- Making assignments one at a time

	1	2	3	4	5
1	M	4	0	3	0
2	4	M	3	0	0
3	0	3	M	1	1
4	3	0	1	M	0
5	0	0	1	0	M

Traveling Salesman Problem - Example

- Making assignments one at a time

	1	2	3	4	5
1	M	4	0	3	0
2	4	M	3	0	0
3	0	3	M	1	1
4	3	0	1	M	0
5	0	0	1	0	M

Traveling Salesman Problem - Example

- Making assignments one at a time

	1	2	3	4	5
1	M	4	0	3	0
2	4	M	3	0	0
3	0	3	M	1	1
4	3	0	1	M	0
5	0	0	1	0	M

Traveling Salesman Problem - Example

- Making assignments one at a time

	1	2	3	4	5
1	M	4	0	3	0
2	4	M	3	0	0
3	0	3	M	1	1
4	3	0	1	M	0
5	0	0	1	0	M

Traveling Salesman Problem - Example

- Making assignments one at a time

	1	2	3	4	5
1	M	4	0	3	0
2	4	M	3	0	0
3	0	3	M	1	1
4	3	0	1	M	0
5	0	0	1	0	M

Traveling Salesman Problem - Example

- Making assignments one at a time

	1	2	3	4	5
1	M	4	0	3	0
2	4	M	3	0	0
3	0	3	M	1	1
4	3	0	1	M	0
5	0	0	1	0	M

Traveling Salesman Problem - Example

- Making assignments one at a time

	1	2	3	4	5
1	M	4	0	3	0
2	4	M	3	0	0
3	0	3	M	1	1
4	3	0	1	M	0
5	0	0	1	0	M

Traveling Salesman Problem - Example

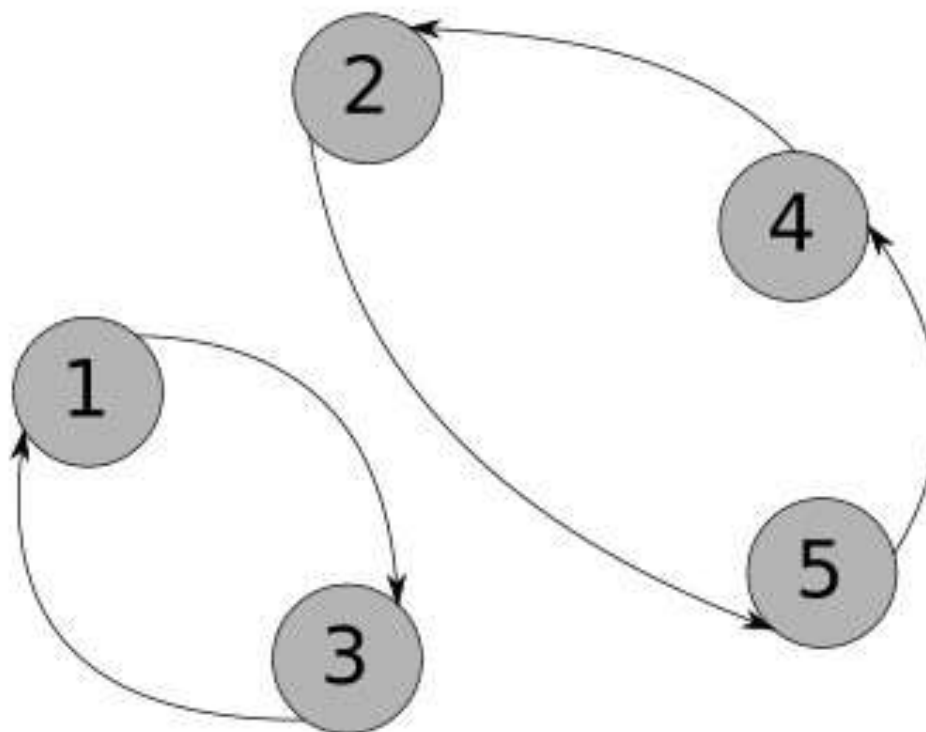
- The optimal solution given by the Hungarian algorithm for the travelling salesman problem is

	1	2	3	4	5
1	M	4	0	3	0
2	4	M	3	0	0
3	0	3	M	1	1
4	3	0	1	M	0
5	0	0	1	0	M

1 → 3 2 → 5 3 → 1 4 → 2 5 → 4

What is wrong with it?

Traveling Salesman Problem – Sub Tours



Sub tour of length 2

Sub tour of length 3

SUB TOURS ARE NOT ALLOWED

Traveling Salesman Problem - Formulation

$$\text{Minimize} \quad \sum_{i=1}^n \sum_{j=1}^n C_{ij} d_{ij}$$

subject to

$$\sum_{i=1}^n X_{ij} = 1 \quad \text{for } j = 1, 2, \dots, n$$

$$\sum_{j=1}^n X_{ij} = 1 \quad \text{for } i = 1, 2, \dots, n$$

$$X_{ij} = (0,1)$$

and additional constraints

Traveling Salesman Problem – Heuristic Algorithms

Nearest Neighborhood Algorithm

- According to this heuristic a good solution of the travelling salesman problem can be obtained by starting with any city and then connecting it with the closest one.
- The just added city is then linked to its nearest unlinked city. Any of the ties are broken arbitrarily.
- This process continues until a tour is formed.

Nearest Neighborhood Algorithm - Example

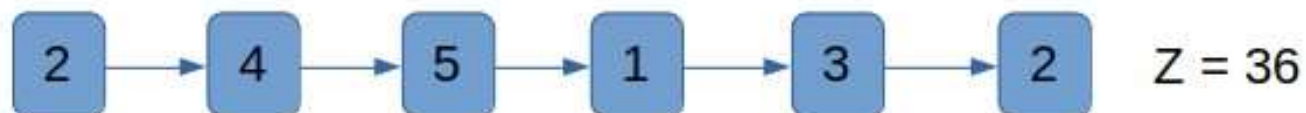
	1	2	3	4	5
1	-	10	8	9	7
2	10	-	10	5	6
3	8	10	-	8	9
4	9	5	8	-	6
5	7	6	9	6	-

Nearest Neighborhood Algorithm - Example

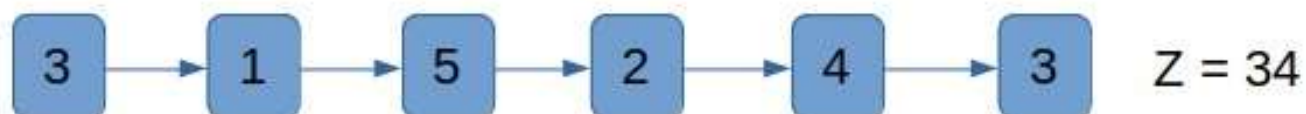
Starting with city 1



Starting with city 2



Starting with city 3



Nearest Neighborhood Algorithm - Example

Starting with city 4



Starting with city 5



The best solution is 1 – 5 – 2 – 4 – 3 – 1 with $Z = 34$

Traveling Salesman Problem – Heuristic Algorithms

Twice Around the Tree Heuristic

- Construct the minimum spanning tree for the given travelling salesman problem.
- Duplicate the edges of the minimum spanning tree to create the Eulerian circuit.
- Delete the repeated edges suitably to obtain feasible solutions.

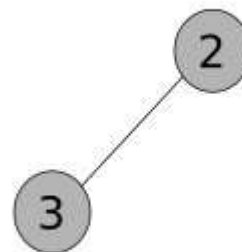
Twice Around The Tree Heuristic - Example

	1	2	3	4	5
1	-	120	220	150	210
2	120	-	100	110	130
3	220	80	-	160	185
4	150	-	160	-	190
5	210	130	185	-	-

Twice Around the Tree Heuristic - Example

Minimum spanning tree for the given travelling salesman problem

	1	2	3	4	5
1	-	120	220	150	210
2	120	-	100	110	130
3	220	80	-	160	185
4	150	-	160	-	190
5	210	130	185	-	-



$$2 - 1 = 120 \quad 3 - 1 = 220$$

$$2 - 4 = 110 \quad 3 - 4 = 160$$

$$2 - 5 = 130 \quad 3 - 5 = 185$$

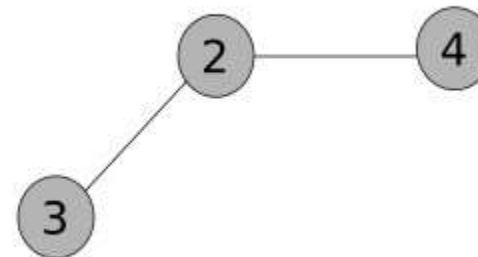
{2, 3}

{1, 4, 5}

Twice Around The Tree Heuristic - Example

Minimum spanning tree for the given travelling salesman problem

	1	2	3	4	5
1	-	120	220	150	210
2	120	-	100	110	130
3	220	80	-	160	185
4	150	-	160	-	190
5	210	130	185	-	-



$$3 - 1 = 220 \quad 4 - 1 = 150$$

$$3 - 5 = 185 \quad 4 - 5 = 190$$

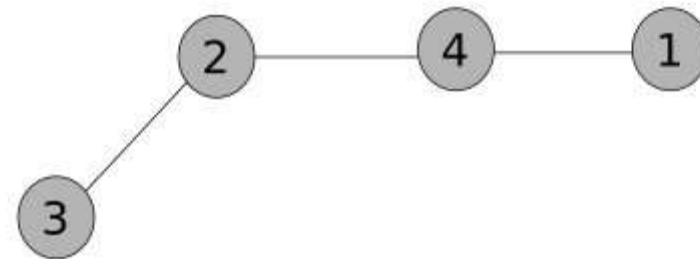
{2, 3, 4}

{1, 5}

Twice Around The Tree Heuristic - Example

Minimum spanning tree for the given travelling salesman problem

	1	2	3	4	5
1	-	120	220	150	210
2	120	-	100	110	130
3	220	80	-	160	185
4	150	-	160	-	190
5	210	130	185	-	-



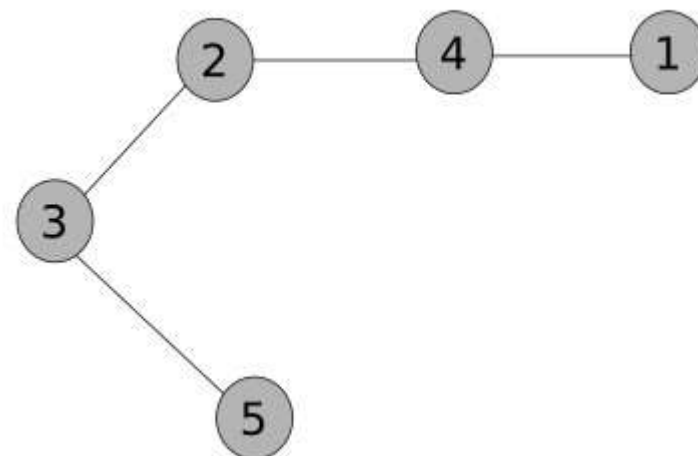
$$3 - 5 = 185 \quad 1 - 5 = 210$$

{2, 3, 4, 1}
{5}

Twice Around The Tree Heuristic - Example

Minimum spanning tree for the given travelling salesman problem

	1	2	3	4	5
1	-	120	220	150	210
2	120	-	100	110	130
3	220	80	-	160	185
4	150	-	160	-	190
5	210	130	185	-	-



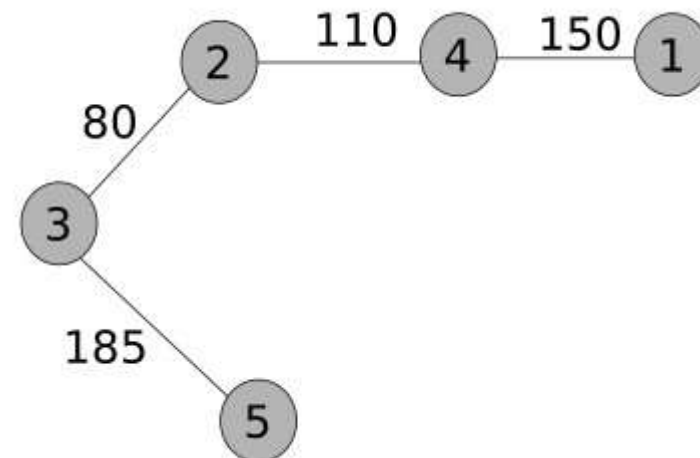
{2, 3, 4, 1, 5}

{ }

Twice Around The Tree Heuristic - Example

Minimum spanning tree for the given travelling salesman problem

	1	2	3	4	5
1	-	120	220	150	210
2	120	-	100	110	130
3	220	80	-	160	185
4	150	-	160	-	190
5	210	130	185	-	-

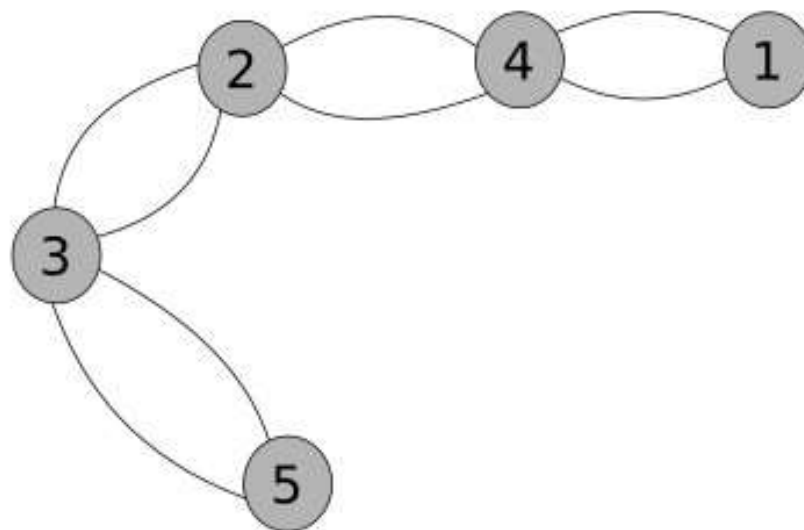


Length of the minimum spanning tree is
 $185 + 80 + 110 + 150 = 525$

{2, 3, 4, 1, 5}
 {}

Twice Around The Tree Heuristic - Example

Duplicating the edges of the minimum spanning tree



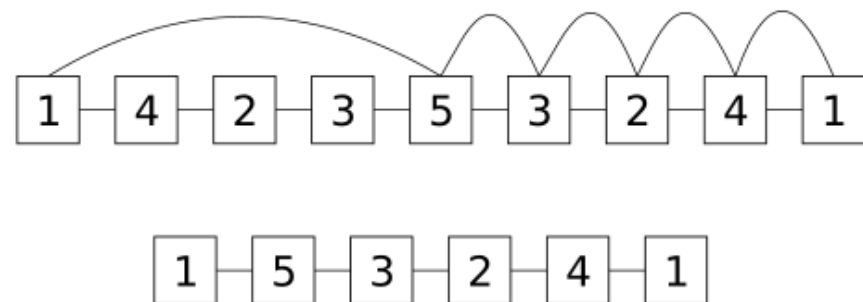
Eulerian circuit is

1 – 4 – 2 – 3 – 5 – 3 – 2 – 4 – 1

Twice Around The Tree Heuristic - Example

Deleting repeated edges to obtain the feasible solutions

	1	2	3	4	5
1	-	120	220	150	210
2	120	-	100	110	130
3	220	80	-	160	185
4	150	-	160	-	190
5	210	130	185	-	-

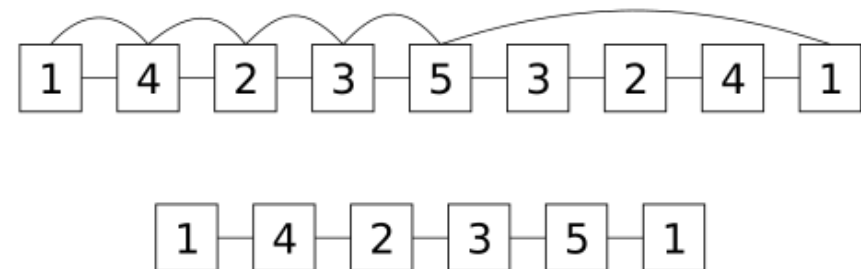


Associated cost, $Z = 210 + 185 + 80 + 110 + 150 = 735$

Twice Around The Tree Heuristic - Example

Deleting repeated edges to obtain the feasible solutions

	1	2	3	4	5
1	-	120	220	150	210
2	120	-	100	110	130
3	220	80	-	160	185
4	150	-	160	-	190
5	210	130	185	-	-

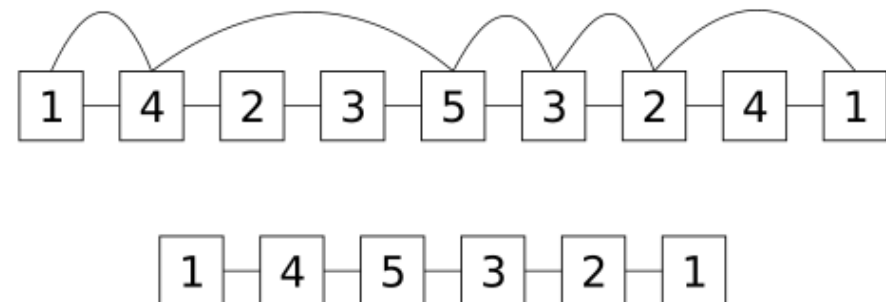


Associated cost, $Z = 150 + - + 100 + 185 + 210 = \text{Not Feasible}$

Twice Around The Tree Heuristic - Example

Deleting repeated edges to obtain the feasible solutions

	1	2	3	4	5
1	-	120	220	150	210
2	120	-	100	110	130
3	220	80	-	160	185
4	150	-	160	-	190
5	210	130	185	-	-

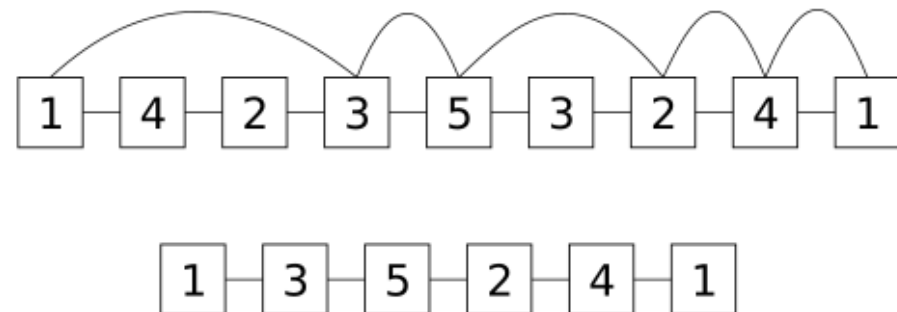


Associated cost, $Z = 150 + 190 + 185 + 80 + 120 = 725$

Twice Around The Tree Heuristic - Example

Deleting repeated edges to obtain the feasible solutions

	1	2	3	4	5
1	-	120	220	150	210
2	120	-	100	110	130
3	220	80	-	160	185
4	150	-	160	-	190
5	210	130	185	-	-

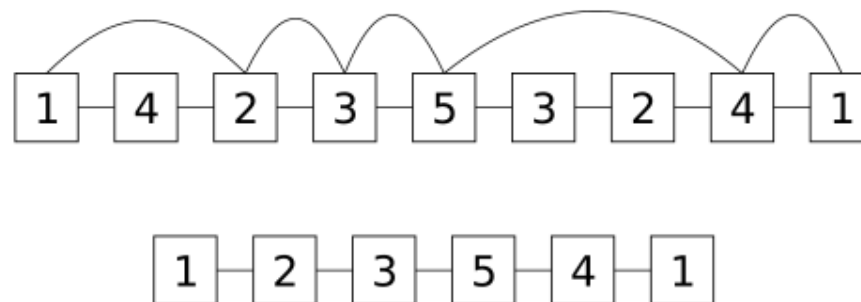


Associated cost, $Z = 220 + 185 + 130 + 110 + 150 = 795$

Twice Around The Tree Heuristic - Example

Deleting repeated edges to obtain the feasible solutions

	1	2	3	4	5
1	-	120	220	150	210
2	120	-	100	110	130
3	220	80	-	160	185
4	150	-	160	-	190
5	210	130	185	-	-

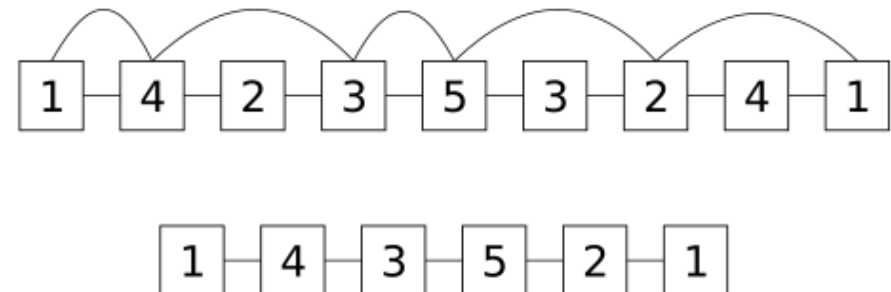


Associated cost, $Z = 120 + 100 + 185 + - + 150 = \text{Not Feasible}$

Twice Around The Tree Heuristic - Example

Deleting repeated edges to obtain the feasible solutions

	1	2	3	4	5
1	-	120	220	150	210
2	120	-	100	110	130
3	220	80	-	160	185
4	150	-	160	-	190
5	210	130	185	-	-



Associated cost, $Z = 150 + 160 + 185 + 130 + 120 = 745$

Twice Around The Tree Heuristic - Example

The best solution is



Associated cost, $Z = 150 + 190 + 185 + 80 + 120 = 725$

Suggested Reading List

1. F S Hillier, G J Liberman, Introduction to Operations Research, 9th edition, McGrawHill publication
2. G. Srinivasan, Operations research – principles and applications, 2nd edition, PHI Learning
3. H A Taha, Operations Research – An Introduction, 8th edition, Pearson Prentice Hall
4. S D Sharma, Operations Research – Theory, Methods and Applications, 5th edition, Kedarnath - Ramnath & Co.

For more course related information, please visit

<https://sites.google.com/view/operationsresearchgiet/home>

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OPERATIONS RESEARCH

UNIT 2

Lecture 5

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Lecture Details:

Sequencing – Introduction - N Jobs Through Two Machines

Operations Research/ME, 5 Semester.

LEARNING OUTCOMES

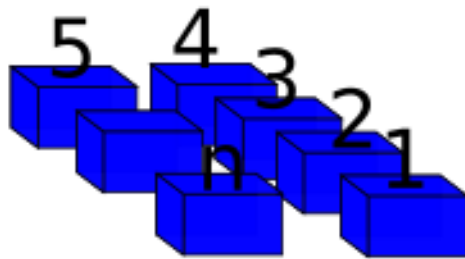
Upon completion of this lecture, student should be able to

- Prepare schedule to process jobs with fixed processing order through two machines
- Develop a solver to tackle flow shop sequencing problems

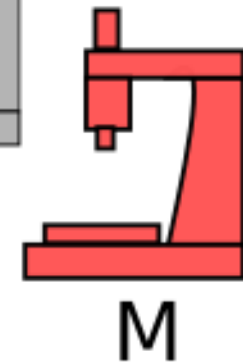
Sequencing - Introduction

- Sequencing problem involves in determining the sequence (order) in which two or more jobs should be processed on one or more machines in order to optimize some measure of effectiveness.
- Constraints placed on the problem are may be due date for each job, processing order of each job on each machine, or variable processing times.
- Measure of effectiveness could be the total elapsed time between the start of the first job on first machine and the completion of the last job on the last machine or maximum tardiness or total tardiness.

Flow Shop Sequencing - Introduction



Job Number	Machine A	Machine B
1	20	4
2	10	12
3	3	5
4	10	8
5	5	6
6	2	12
7	8	4
8	7	10
9	3	6
10	4	1



Flow Shop Sequencing - Introduction

No Passing Rule:

When a job is finished on machine 1, it goes directly on machine 2 if it is empty; otherwise, it starts a waiting line or joins the end of the waiting line if one already exists. Jobs which form a waiting line are processed on machine 2 in the same order that they joined the waiting line.

Lateness:

The difference between the actual completion time of the job and its due date. If the lateness is positive, it is termed as **tardiness**.

Slack Time: Due date minus processing time of a job.

Sequencing/Dispatching Rules

Shortest Processing Time (SPT) Rule

The job with least processing time is picked up first, followed by the one with the next smallest processing time and so on.

Earliest Due Date (EDD) Rule

Jobs are sequenced in the order of the non decreasing due dates.

Slack Time Remaining (STR) Rule

Jobs with the least slack time picked first, followed by the one with the next smallest slack time.

Processing n Jobs Through Two Machines

Johnson's Algorithm

Step 1

Scan all the time periods for the shortest one.

Step 2

If it is for the first machine, place the corresponding item first in the sequence. If it is for the second machine, place the corresponding item last. Cross off both the times for that item

Step 3

Repeat the steps on the remaining.

In case of ties, order the item with the smallest subscript first.

In case of ties between both the machines, order the item according to A .

Processing n Jobs Through Two Machines

Example

A machine operator has to perform operations on two machines for various jobs and respective times are given. Determine the order in which the jobs should be processed in order to minimize the total time required to process all the jobs.

Job Number	Machine A	Machine B
1	20	4
2	10	12
3	3	5
4	10	8
5	5	6
6	2	12
7	8	4
8	7	10
9	3	6
10	4	1

Also find the total processing time and idle times for both the operations.

Processing n Jobs Through Two Machines

Example

Job Number	Machine A	Machine B
1	20	4
2	10	12
3	3	5
4	10	8
5	5	6
6	2	12
7	8	4
8	7	10
9	3	6
10	4	1

									10

Processing n Jobs Through Two Machines

Example

Job Number	Machine A	Machine B
1	20	4
2	10	12
3	3	5
4	10	8
5	5	6
6	2	12
7	8	4
8	7	10
9	3	6

6									10

Processing n Jobs Through Two Machines

Example

Job Number	Machine A	Machine B
1	20	4
2	10	12
3	3	5
4	10	8
5	5	6
7	8	4
8	7	10
9	3	6

6	3	9							10

Processing n Jobs Through Two Machines

Example

Job Number	Machine A	Machine B
1	20	4
2	10	12
4	10	8
5	5	6
7	8	4
8	7	10

6	3	9					7	1	10

Processing n Jobs Through Two Machines

Example

Job Number	Machine A	Machine B
2	10	12
4	10	8
5	5	6
8	7	10

6	3	9	5				7	1	10

Processing n Jobs Through Two Machines

Example

Job Number	Machine A	Machine B
2	10	12
4	10	8
8	7	10

6	3	9	5	8			7	1	10

Processing n Jobs Through Two Machines

Example

Job Number	Machine A	Machine B
2	10	12
4	10	8

6	3	9	5	8	2	4	7	1	10

Processing n Jobs Through Two Machines

Example

Job Number	Machine A	Machine B
1	20	4
2	10	12
3	3	5
4	10	8
5	5	6
6	2	12
7	8	4
8	7	10
9	3	6
10	4	1

Job	Machine A		Machine B	
	Time in	Time out	Time in	Time out
6	0	2	2	14
3	2	5	14	19
9	5	8	19	25
5	8	13	25	31
8	13	20	31	41
2	20	30	41	53
4	30	40	53	61
7	40	48	61	65
1	48	68	68	72
10	68	72	72	73

Idle Time

Total Processing Time
73 Min

Machine A - 1 Min
Machine B - 5 Min

Suggested Reading List

1. A M Natarajan, P Balasubramanei, A Tamilarasi, Operations Research, 1st edition, Pearson.
2. J K Sharma, Operations Research – Problems and Solutions, 3rd edition, MacMillan Publishers India.
3. S D Sharma, Operations Research – Theory, Methods and Applications, 5th edition, Kedarnath - Ramnath & Co.
4. Billy E Gillett, Introduction to Operations Research – A Computer Oriented Algorithmic Approach, 1st edition, McGrawHill Education.

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OPERATIONS RESEARCH

UNIT 2

Lecture 6

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Lecture Details:

Sequencing – N Jobs Through Three Machines

Operations Research/ME, 5 Semester.

LEARNING OUTCOMES

Upon completion of this lecture, student should be able to

- Prepare schedule to process jobs with fixed processing order through three machines
- Develop a solver to tackle flow shop sequencing problems

Flow Shop Sequencing N Jobs Through Three Machines

For the flow shop sequencing involving three machines A, B and C (stages), optimal solution can be found analogously to the two machine problem upon satisfying the either one or both of the following conditions.

$$\min A_i \geq \max B_j$$

$$\min C_i \geq \max B_j$$

Formulate the given three machine (Machines A, B and C) problem as two machine problem (Machines G and H) by following manipulations

$$G_i = A_i + B_i$$

$$H_j = B_j + C_j$$

Flow Shop Sequencing N Jobs Through Three Machines Example

Times required to process various jobs on three machines in order is given below. Determine the optimum sequence to minimize the total processing time. Also find the idle times for the three Machines.

Job	Machine A	Machine B	Machine C
1	3	8	13
2	12	6	14
3	5	4	9
4	2	6	12
5	9	3	8
6	11	1	13

Flow Shop Sequencing N Jobs Through Three Machines Example

Job	Machine A	Machine B	Machine C
1	3	8	13
2	12	6	14
3	5	4	9
4	2	6	12
5	9	3	8
6	11	1	13

Min(A)	2
Max(B)	8
Min(C)	8

$\text{Min } A_i < \text{Max } B_i$ Violated

$\text{Min } C_j = \text{Max } B_j$ Satisfied

Flow Shop Sequencing N Jobs Through Three Machines Example

Job	Machine A	Machine B	Machine C
1	3	8	13
2	12	6	14
3	5	4	9
4	2	6	12
5	9	3	8
6	11	1	13

Job	Machine G	Machine H
	(A + B)	(B + C)
1	11	21
2	18	20
3	9	13
4	8	18
5	12	11
6	12	14

Flow Shop Sequencing N Jobs Through Three Machines Example

Job	Machine G	Machine H
	(A + B)	(B + C)
1	11	21
2	18	20
3	9	13
4	8	18
5	12	11
6	12	14

4					

Flow Shop Sequencing N Jobs Through Three Machines Example

Job	Machine G	Machine H
	(A + B)	(B + C)
1	11	21
2	18	20
3	9	13
5	12	11
6	12	14

4	3				

Flow Shop Sequencing N Jobs Through Three Machines Example

Job	Machine G	Machine H
	(A + B)	(B + C)
1	11	21
2	18	20
5	12	11
6	12	14

4	3	1			5

Flow Shop Sequencing N Jobs Through Three Machines Example

Job	Machine G	Machine H
	(A + B)	(B + C)
2	18	20
6	12	14

4	3	1	6	2	5

Flow Shop Sequencing N Jobs Through Three Machines Example

Job	Machine A	Machine B	Machine C
1	3	8	13
2	12	6	14
3	5	4	9
4	2	6	12
5	9	3	8
6	11	1	13

Job	Machine A		Machine B		Machine C	
	Time in	Time out	Time in	Time out	Time in	Time out
4	0	2	2	8	8	20
3	2	7	8	12	20	29
1	7	10	12	20	29	42
6	10	21	21	22	42	55
2	21	33	33	39	55	69
5	33	42	42	45	69	77

Flow Shop Sequencing N Jobs Through Three Machines Example

Job	Machine A		Machine B		Machine C	
	Time in	Time out	Time in	Time out	Time in	Time out
4	0	2	2	8	8	20
3	2	7	8	12	20	29
1	7	10	12	20	29	42
6	10	21	21	22	42	55
2	21	33	33	39	55	69
5	33	42	42	45	69	77

Total Elapsed Time
77 Minutes

Idle Times
Machine A - 35 Minutes
Machine B - 49 Minutes
Machine C - 08 Minutes

Suggested Reading List

1. A M Natarajan, P Balasubramanei, A Tamilarasi, Operations Research, 1st edition, Pearson.
2. J K Sharma, Operations Research – Problems and Solutions, 3rd edition, MacMillan Publishers India.
3. S D Sharma, Operations Research – Theory, Methods and Applications, 5th edition, Kedarnath - Ramnath & Co.
4. Billy E Gillett, Introduction to Operations Research – A Computer Oriented Algorithmic Approach, 1st edition, McGrawHill Education.

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OPERATIONS RESEARCH

UNIT 2

Lecture 7

~~Presented By:~~

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Lecture Details:

Job Shop Sequencing – Two Jobs Through M Machines

Operations Research/ME, 5 Semester.

LEARNING OUTCOMES

Upon completion of this lecture, student should be able to

- Distinguish flow shop and job shop sequencing problems
- Prepare job shop schedule to process two jobs on M machines

Job Shop Sequencing Introduction

In job shop scheduling, multiple jobs are to be processed on several machines. Sequence of processing of each job may or may not be the same, but it must be performed in the given order. Each task is performed on specific machine. The objective is to process the jobs in the given sequence on specific machines in such a way that the total elapsed time is minimized.

Assumptions for the job shop sequencing problem are

- Once started, jobs must run to completion.
- No machine processes more than one job at a time.
- Machine takes the next job, only upon the completion of the present job.

Job Shop Sequencing

Two Jobs Through M Machines

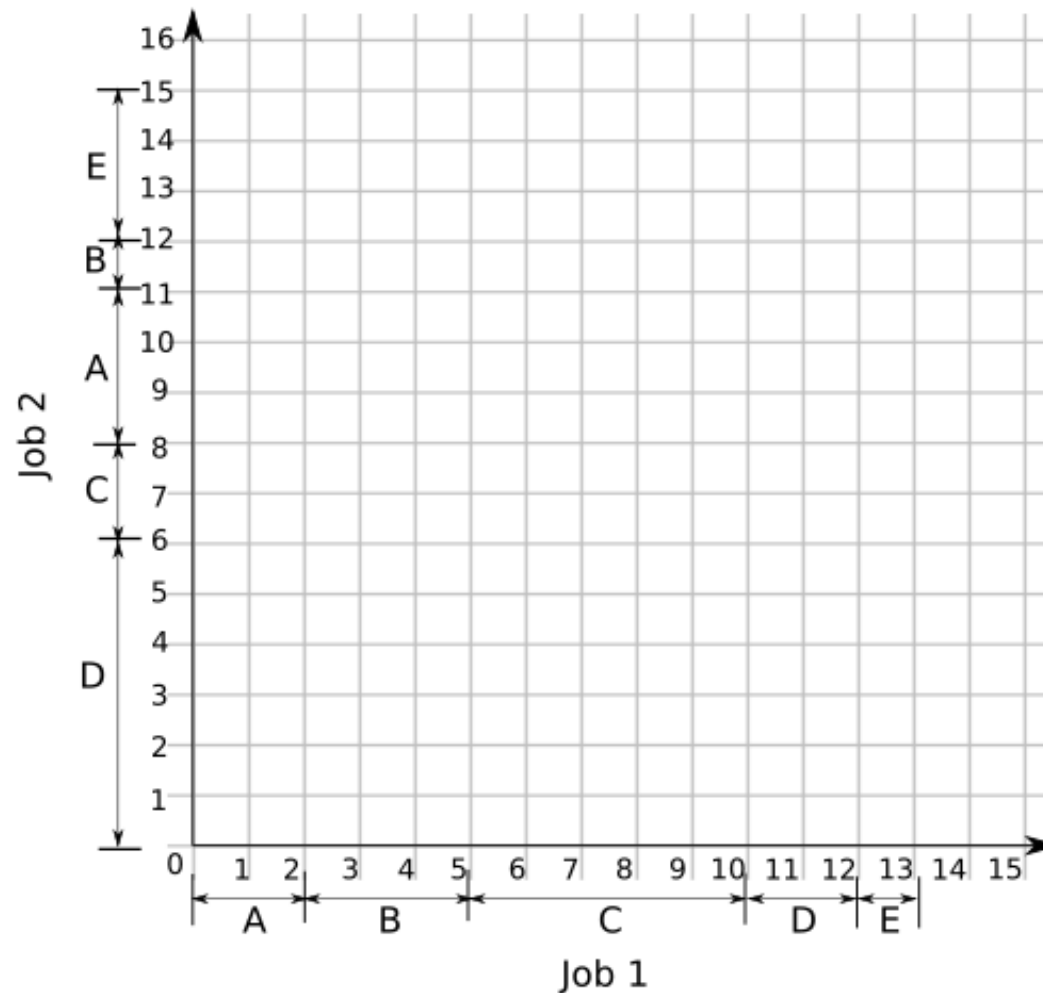
Example

Use graphical method to minimize the time required to process the following jobs on the machines. For each machine specify the job which should be done first. Also calculate the total elapsed time to complete both jobs.

Job 1	Sequence	A	B	C	D	E
	Time(hr)	2	3	5	2	1
Job 2	Sequence	D	C	A	B	E
	Time(hr)	6	2	3	1	3

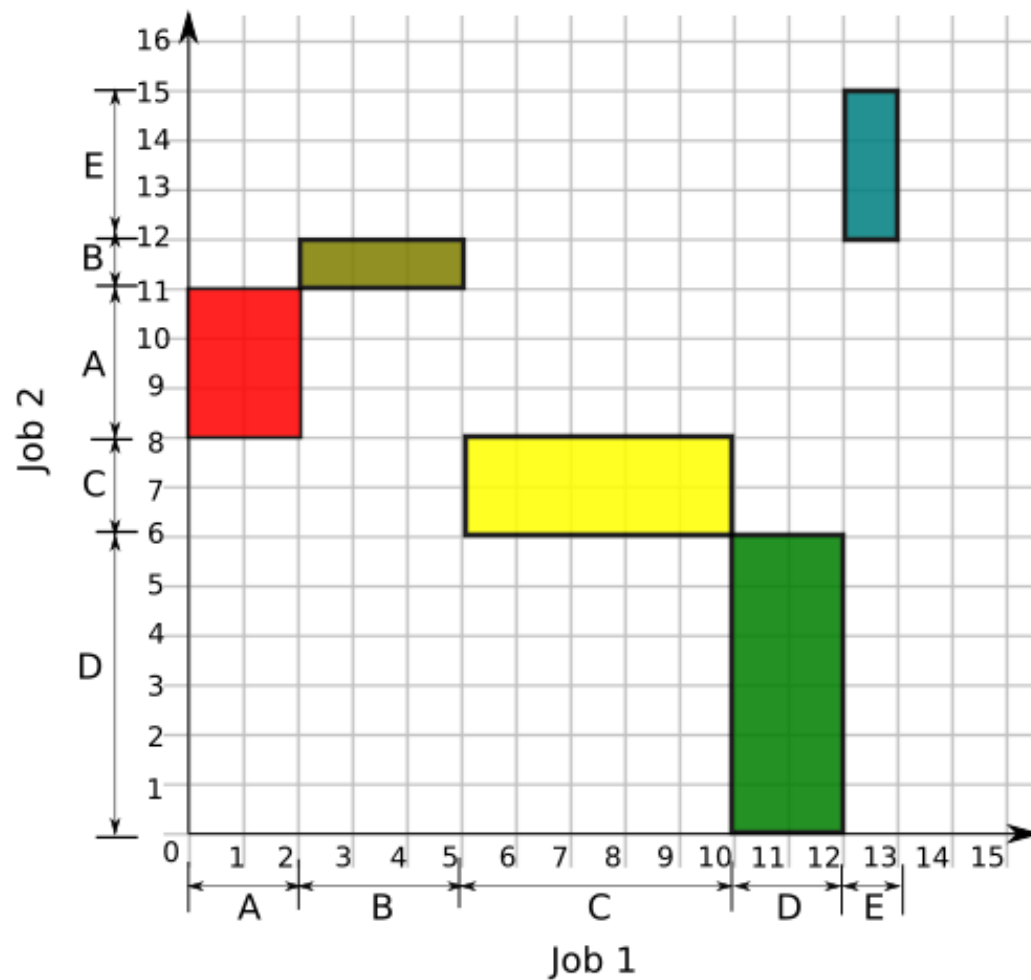
Job Shop Sequencing

Two Jobs Through M Machines



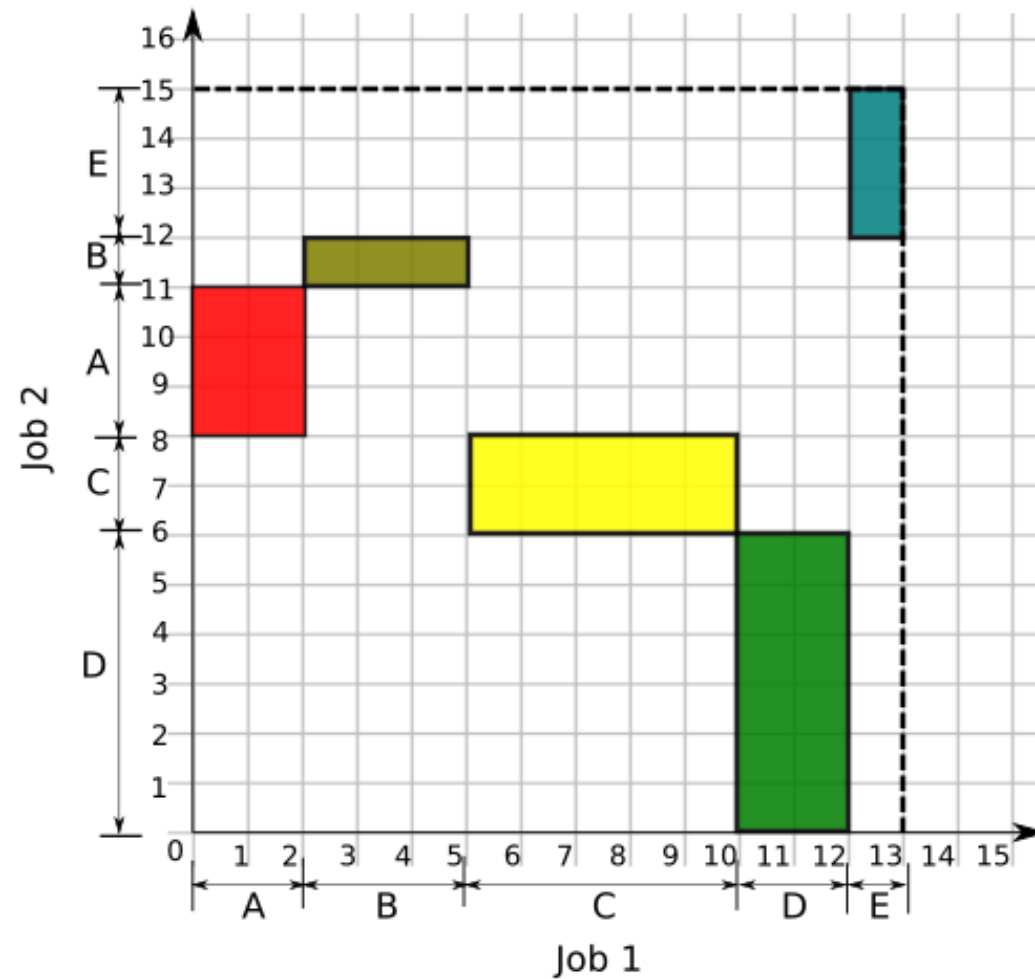
Job Shop Sequencing

Two Jobs Through M Machines



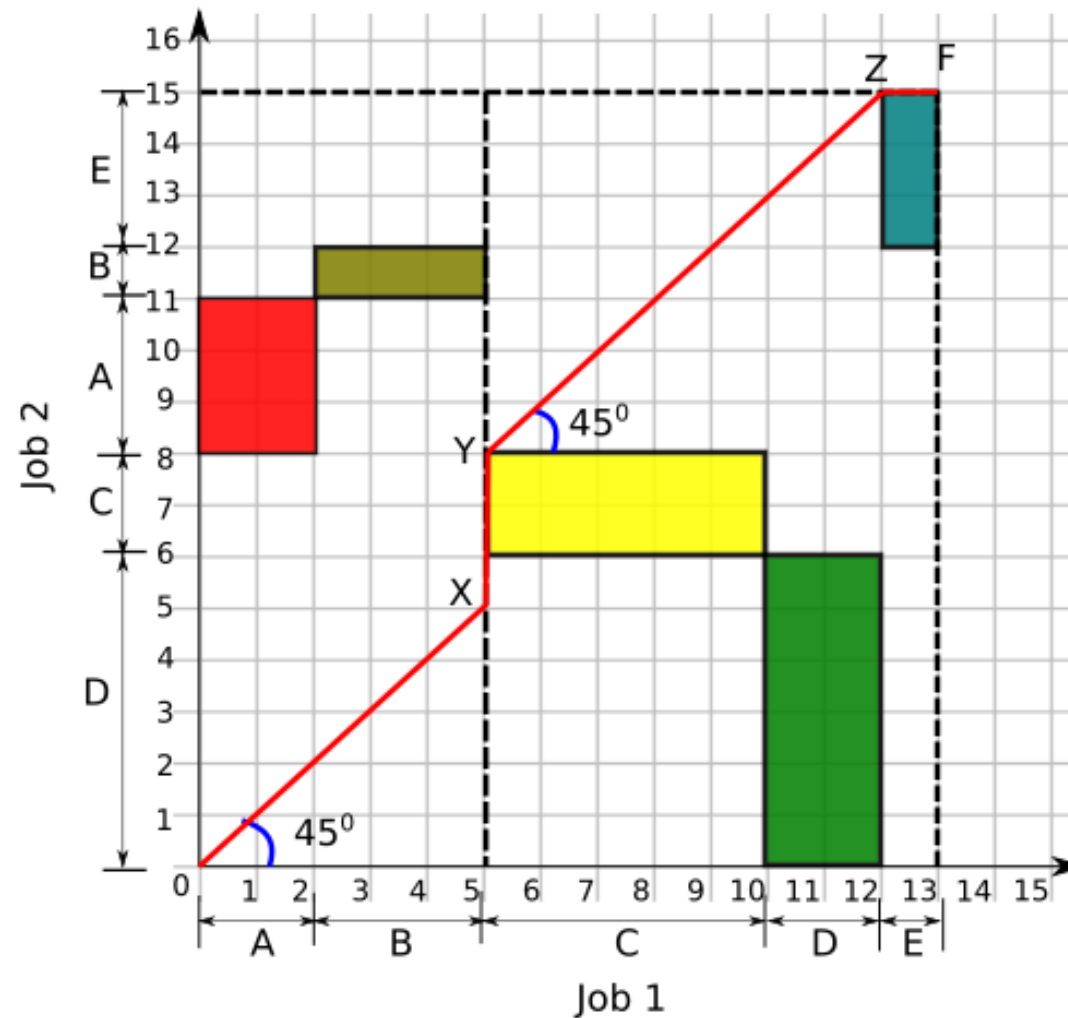
Job Shop Sequencing

Two Jobs Through M Machines



Job Shop Sequencing

Two Jobs Through M Machines

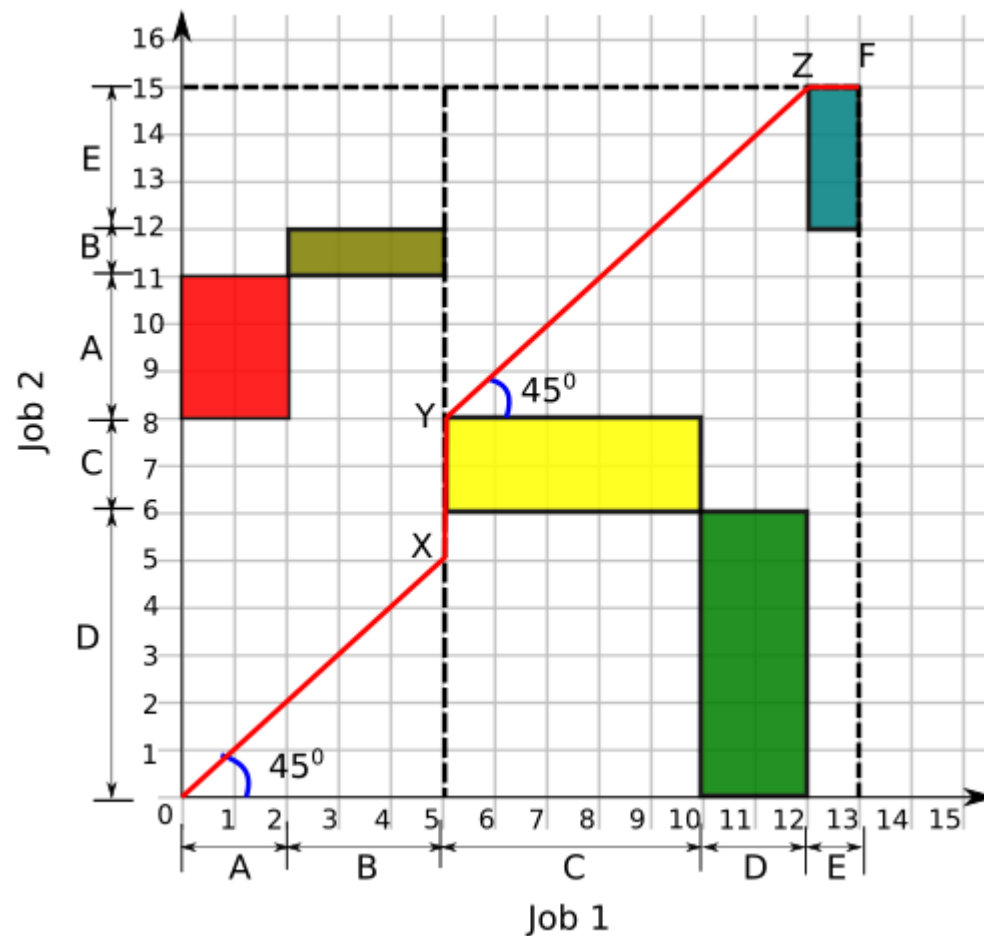


Job Shop Sequencing

Two Jobs Through M Machines

Idle Time for Job 1
3 Hours

Idle Time for Job 2
1 Hour



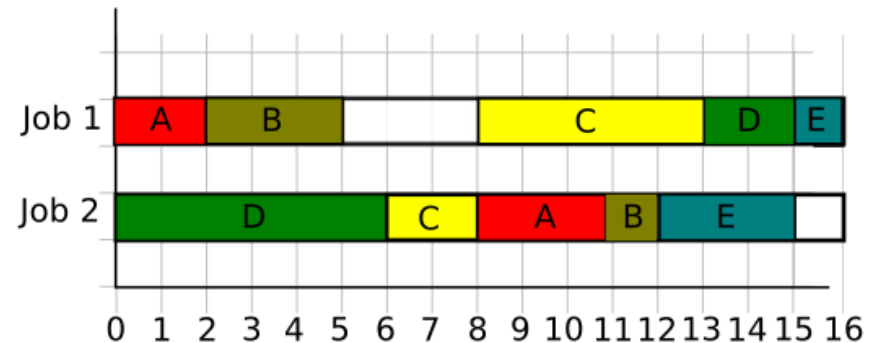
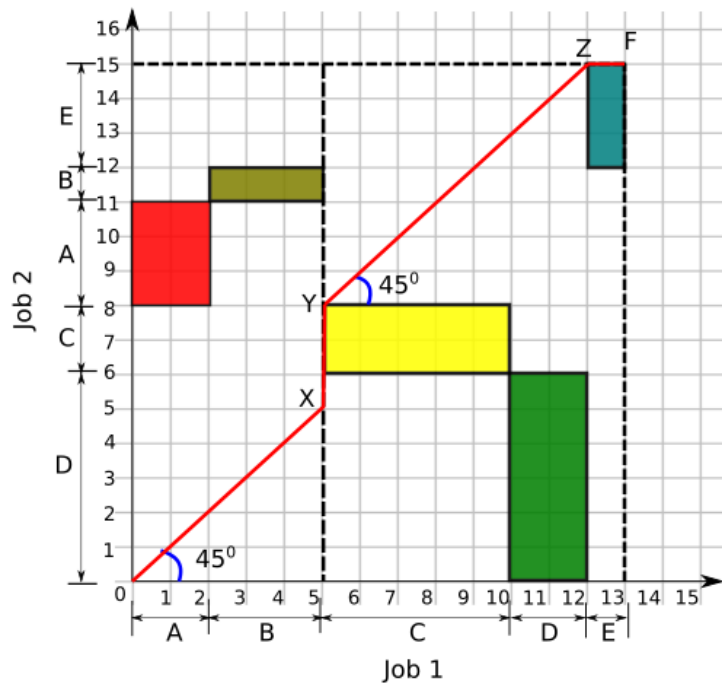
Total Elapsed Time

$13 + 3 = 16$ hours (Job 1)

$15 + 1 = 16$ hours (Job 2)

Job Shop Sequencing

Two Jobs Through M Machines



The optimal sequence

Machine A – Job 1 precedes Job 2

Machine B – Job 1 precedes Job 2

Machine C – Job 2 precedes Job 1

Machine D – Job 2 precedes Job 1

Machine E – Job 2 precedes Job 1

Suggested Reading List

1. A M Natarajan, P Balasubramaneai, A Tamilarasi, Operations Research, 1st edition, Pearson.
2. J K Sharma, Operations Research – Problems and Solutions, 3rd edition, MacMillan Publishers India.
3. S D Sharma, Operations Research – Theory, Methods and Applications, 5th edition, Kedarnath - Ramnath & Co.
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OPERATIONS RESEARCH

UNIT 2

Lecture 8

~~Presented By:~~



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Lecture Details:

Replacement Theory

Operations Research/ME, 5 Semester.

LEARNING OUTCOMES

Upon completion of this lecture, student should be able to

- Explain necessity of replacement of items
- Describe various types of failure of items

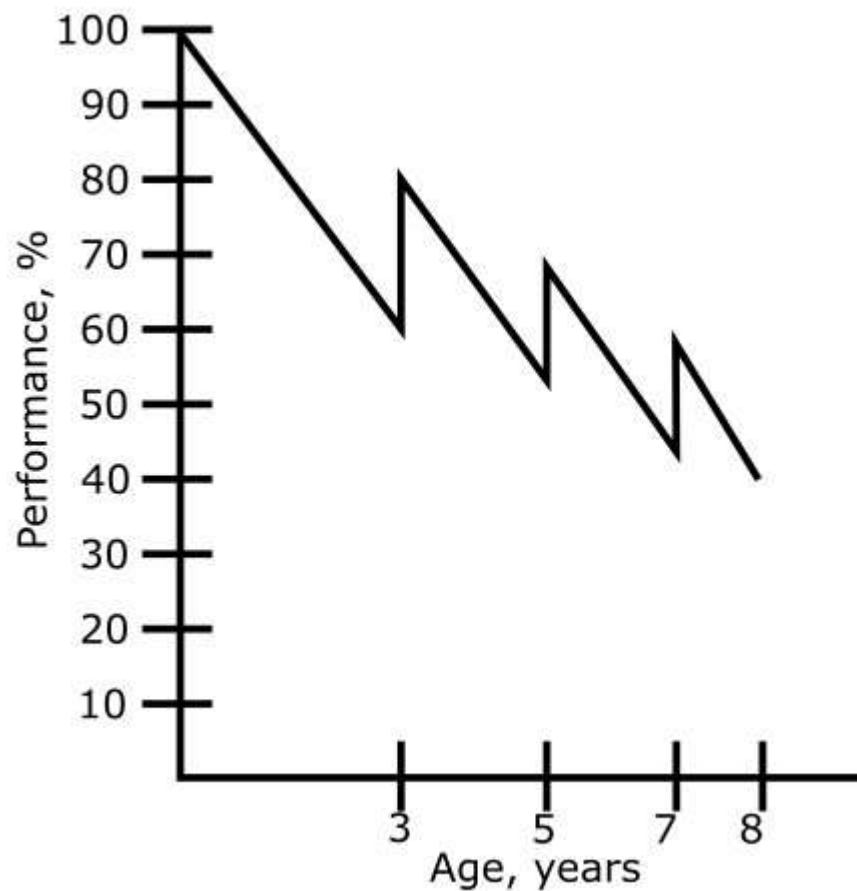
INTRODUCTION

- The problem of replacement is felt when the job performing units such as men, machines, parts etc., become less effective or useless due to either sudden or gradual deterioration in their efficiency, failure or breakdown.
- Regular maintenance activity is carried out to bring up the performance by replacing the worn out parts or failed parts, repair or overhaul.
- Maintenance activity may not bring the working efficiency to the previous level.
- After few maintenance activities, it is not possible to improve the efficiency and it may cause reduced production rates, increased production costs etc.

INTRODUCTION



Vertical Milling Center



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INTRODUCTION

- When maintenance and overhead costs becomes excessive, replacement is warranted to reduce the costs.
- In case of vehicles/production machines (item that fails gradually with time), the replacement decision may be taken to balance the increasing maintenance cost with the decreasing money value of the item as time passes.
- Replacement is necessary because of the following reasons
 - ✓ Obsolescence
 - ✓ Production requirements
 - ✓ Increased maintenance costs
 - ✓ Equipment failure

INTRODUCTION



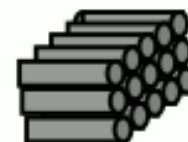
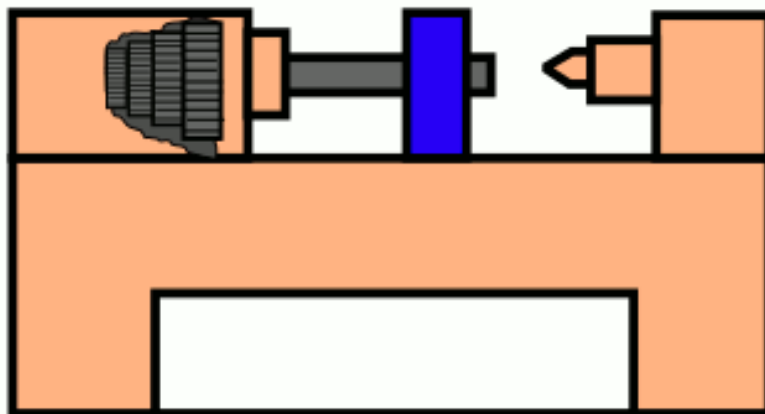
- Items such as light bulbs, computers etc., do not give any indication of deterioration with time but fail all of a sudden and become completely useless.
- The replacement decision is to balance the wasted life of items replaced before failure against the costs incurred when items fail in service.



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TYPES OF FAILURE

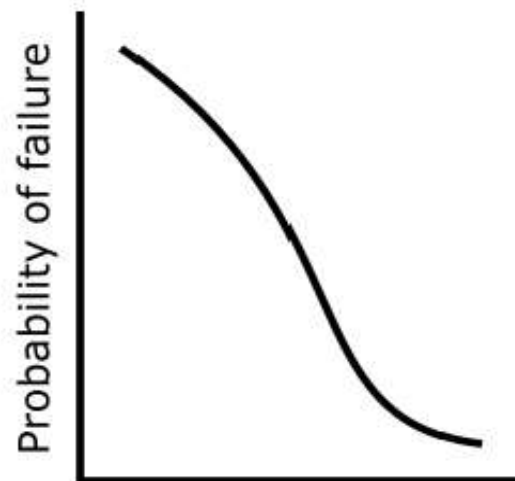
- **Gradual failure** – As the lifetime of the item increases, its operating efficiency decreases causing increased maintenance and operating costs, decreased productivity and decrease in the resale value



TYPES OF FAILURE

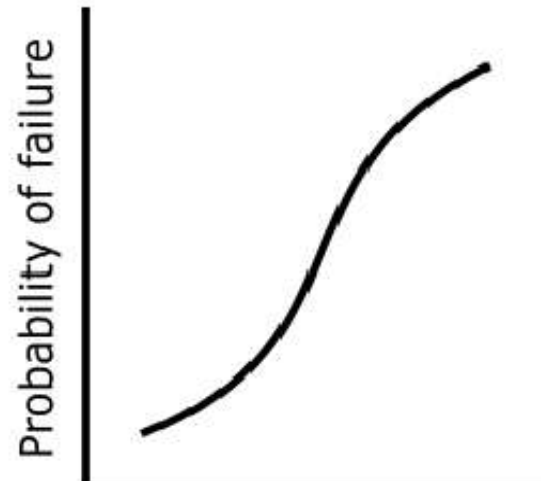
- **Sudden failure** – Failure of an item occurring after some period of service rather than deterioration while in service. The period of giving desired service is not constant but follows some frequency distribution which may be progressive, retrogressive or random in nature.
- ✓ **Progressive failure** – Probability of failure increases with increase in its life
- ✓ **Retrogressive failure** – Probability of failure in the beginning of life of an item is more and due to change of time, the chances of failure decreases.
- ✓ **Random failure** - The constant probability of failure is associated with items that fail from random causes like physical shocks not related to age.

TYPES OF FAILURE



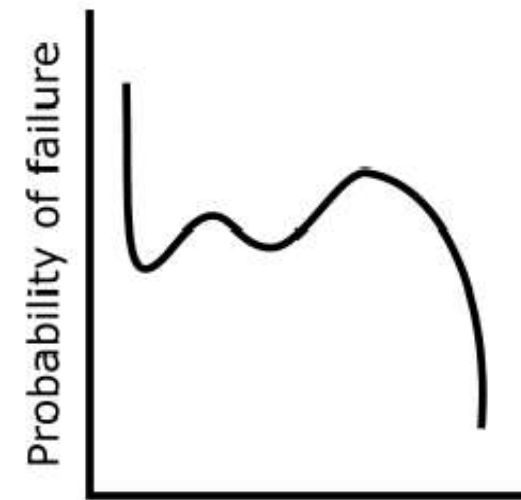
Age of the Item

Progressive failure



Age of the Item

Retrogressive failure
Sudden failure of an item



Age of the Item

Random failure

Suggested Reading List

1. A M Natarajan, P Balasubramaneai, A Tamilarasi, Operations Research, 1st edition, Pearson.
2. J K Sharma, Operations Research – Problems and Solutions, 3rd edition, MacMillan Publishers India.
3. S D Sharma, Operations Research – Theory, Methods and Applications, 5th edition, Kedarnath - Ramnath & Co.
4. H A Taha, Operations Research – An Introduction, 8th edition, Pearson Prentice Hall

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OPERATIONS RESEARCH

UNIT 3



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Lecture Details:

Game Theory

Operations Research/ME, 5 Semester.

LEARNING OUTCOMES

Upon completion of this lecture, student should be able to

- Explain basic concepts of game theory
- Apply concepts of game theory to arrive at a decision in competitive situations

Introduction

- Game theory is a type of decision theory in which one's choice of action is determined after taking into account all possible alternatives available to an opponent playing the same game, rather than just by the possibilities of several outcomes.
- Jon Van Neumann is considered the father of game theory.
- Game is defined as an activity between two or more persons involving a course of action by each person according to a set of rules, at the end of which each person receives some benefit or satisfaction or suffers loss (negative benefit).
- The set of rules defines the game. Following these set of rules, each of the participant chooses one of the courses of action, and the play is said to occur.

Characteristics of Game Theory

Different game models exist based on the characteristics.

- Strategies - The strategy for a player is the list of all possible actions that are taken for every pay-off that might arise.
- Number of persons - If a game involves only two players, then it is called as a two person game. If the number of players are more than two, the game is known as n-person game.
- Number of strategies - If the number of strategies (activities) is finite, then the game is a finite game, if not, it is called infinite game.
- A quantitative measure of satisfaction a person gets at the end of each play is called as payoff.

Characteristics of Game Theory

- Every combination of course of action determines an outcome which results in gains to the participants. The gain may be positive, negative or zero.
- The gains (pay-off) for each and every play are fixed and specified in advance and are known to each player.
- The choices are assumed to be made simultaneously so that no player knows the choices made by others until he/she has decided his own.

Basic Terminology

- Zero-Sum games - If the players make payments only to each other, i.e., the loss of one is the gain of another - nothing comes from outside.
- A game which is not zero sum is called non zero-sum game.
- Pure strategies - If a player knows exactly what the other player is going to do, a deterministic situation arises, and he/she choose one particular strategy with the objective of maximizing the gain.
- Mixed strategies - When both players are guessing as to which course of action is to be selected on a particular occasion with some fixed probability, it is a mixed strategy game. Thus there is a probabilistic situation and objective of the players is to maximize gains.

Two-Person Zero-Sum Game

- A game with only two persons is said to be two person zero sum game if the gain of one player is equal to the loss of the other.
- The pay-off in terms of gains or losses, when players selected their particular strategies, can be represented in the form of a matrix, called the pay-off matrix.

Two-Person Zero-Sum Game

Suppose the player A has m activities and the player B has n activities. Then the pay-off matrix can be formed by adopting the following rules.

- Row designations for each matrix are activities available to player A.
- Column designations for each matrix are activities available to player B.
- Cell entry v_{ij} is the payment to player A in A's pay-off matrix, when A chooses the activity i and B chooses activity j .
- With a zero sum two person game, the cell entry in the player B's payoff matrix is negative of the corresponding entry in A's payoff matrix - so sum of payoff matrices of both the players is ultimately zero.

Two-Person Zero-Sum Game

	B_1	B_2	...	B_n
A_1	v_{11}	v_{12}	...	v_{1n}
A_2	v_{21}	v_{22}	...	v_{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_m	v_{m1}	v_{m2}	...	v_{mn}

Player A's pay-off matrix

	B_1	B_2	...	B_n
A_1	$-v_{11}$	$-v_{12}$	...	$-v_{1n}$
A_2	$-v_{21}$	$-v_{22}$	...	$-v_{2n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_m	$-v_{m1}$	$-v_{m2}$	...	$-v_{mn}$

Player B's pay-off matrix

Minimax(Maximin) criterion and Optimal Strategies

The minimax criterion of optimality states that if a player lists the worst possible outcomes of all his potential strategies, he will choose the strategies that are most suitable for him, which corresponds to the best of the worst outcomes. Such a strategy is called an optimal strategy.

		B			
		I	II	III	
A	I	-3	-2	6	-3
	II	2	0	2	0
	III	5	-2	-4	-4
		5	0	6	
					Minimax v_{II_2}

Maximin v_{II}

$$\text{Max}\{-3, 0, -4\} = 0$$

$$\text{Min}\{5, 0, 6\} = 0$$

Note

$$v_{II} = v_{II_2} = 0$$

Maximin = Minimax

Saddle point

Two-Person Zero-Sum Game

- **Saddle point** - A saddle point of a pay-off matrix is the position of such an element in the pay-off matrix which is minimum in its row and maximum in its column.
- **Optimal strategies** - If the pay-off matrix has the saddle point (r, s) then the player (A and B) are said to have r^{th} and s^{th} optimal strategies respectively.
- **Value of the game** - The payoff (v_{ij}) at the saddle point (r, s) is called the value of the game, and it is obviously equal to its maximin(v) and minimax value(v_{Π_2}) of the game.
 - ❖ A game is said to be fair game, if $v_{\Pi_2} = v_{\Pi_1} = 0$.
 - ❖ A game is said to be strictly determinable, if $v_{\Pi_2} = v_{\Pi_1} = v$.

Games with Saddle Point

Solve the following game, whose pay-off matrix is given as

		Player B				
		I	II	III	IV	V
Player A	I	-2	0	0	5	3
	II	3	2	1	2	2
	III	-4	-3	0	-2	6
	IV	5	3	-4	2	-6

Games with Saddle Point

Step 1: Identify the minimum of each row, and enclose it in a rectangle.

		Player B					Row min
		I	II	III	IV	V	
Player A	I	-2	0	0	5	3	-2
	II	3	2	1	2	2	1
	III	-4	-3	0	-2	6	-4
	IV	5	3	-4	2	-6	-6

Games with Saddle Point

Step 2: Select the maximum element in each column and enclose it in a circle.

		Player B					Row min
		I	II	III	IV	V	
Player A	I.	-2	0	0	5	3	-2
	II.	3	2	1	2	2	1
	III.	-4	-3	0	-2	6	-4
	IV.	5	3	-4	2	-6	-6
Col max		5	3	1	5	6	

Games with Saddle Point

Step 3: Find out the element which is enclosed by the rectangle as well as the circle. Such entry is the value of the game and that position is called as the saddle point

		Player B						
		I	II	III	IV	V		
Player A	I	-2	0	0	5	3	Maximin(v) = 1	Minimax(w) = 1
	II	3	2	1	2	2		
	III	-4	-3	0	-2	6		
	IV	5	3	-4	2	-6		

Games with Saddle Point

		Player B				
		I	II	III	IV	V
Player A	I	-2	0	0	5	3
	II	3	2	1	2	2
	III	-4	-3	0	-2	6
	IV	5	3	-4	2	-6

Maximin(v_1) =
1

Minimax(v_2) = 1

The best strategy for player A is II and player B is III
The value of the game is 1 to player A and -1 to player B

Games with Saddle Point

Solve the following game, whose pay-off matrix is given as

		Player B		
		I	II	III
Player A	I	-2	15	-2
	II	-5	-6	-4
	III	-5	20	-8

Games with Saddle Point

Selecting row minimum and column maximum, check if there is a saddle point.

		Player B			Row min
		I	II	III	
Player A	I	-2	15	-2	-2
	II	-5	-6	-4	
	III	-5	20	-8	
Col max		-2	20	-2	-2

- The game has two saddle points in position (1,1) and (1,3).
- The best strategy for player A is I
- The best strategy for player B is I or III.
- The value of the game is -2 to player A and 2 to player B

Games Without Saddle Point

- When there is no saddle point exists, then both the players must determine an optimal combination of strategies to find a saddle (equilibrium) point.
- The optimal strategy combination for each player may be determined by assigning to each strategy its probability of being chosen.
- The strategies so determined are called mixed strategies because they are probabilistic combination of available choices of strategy.
- The value of game obtained by the use of mixed strategies represents least pay-off which player A can expect to win and the least which player B can lose.

Games Without Saddle Point

- The expected pay-off to a player in a game with arbitrary pay-off matrix $[v_{ij}]$ of order $m \times n$ is defined as

$$V = \sum_{i=1}^m \sum_{j=1}^n p_i v_{ij} q_j$$

here

$(p_1, p_2, \dots, p_m)$ are mixed strategies of player A

$(q_1, q_2, \dots, q_n)$ are mixed strategies of player B

- A particular strategy with a particular probability a player chooses can also be interpreted as the relative frequency with which a strategy is chosen from the number of strategies of the game.

2 X 2 Games - Mixed Strategies Arithmetic Method

Arithmetic method (odds method or short cut method)

- Subtract the two digits in column 1 and write the difference under column 2, ignoring the sign.
- Subtract the two digits in column 2 and write the difference under column 1, ignoring sign.
- Similarly proceed for the two rows.

The values thus obtained are called as oddments. These are the frequencies with which the players must use their courses of action in their optimum strategies.

2 X 2 Games - Mixed Strategies

Arithmetic Method - Example Problem

Problem:

Two players A and B showing each other, put on a table a coin, with head or tail up. A wins ₹8 when both the coins shows head and ₹1 when both are tails. B wins ₹3 when the coins do not match. Given the choice of being matching player (A) or non matching player (B) which one would you choose and what would be your strategy?

2 X 2 Games - Mixed Strategies Arithmetic Method - Example Problem

The pay-off matrix for player A is

		B	
		H	T
A	H	8	-3
	T	-3	1

2 X 2 Games - Mixed Strategies Arithmetic Method - Example Problem

Checking whether saddle point exists or not

		B		
		H	T	Row min
A	H	8	-3	-3
	T	-3	1	-3
Col max		8	1	

Maximin $\alpha = -3$

Since the saddle point does not exist, the optimal strategies will be mixed strategies.

Minimax $\beta = 1$

2 X 2 Games - Mixed Strategies Arithmetic Method - Example Problem

Taking the difference of two numbers in a column I (II), and put it under the column II (I), ignoring sign.

		B	
		H	T
A	H	8	-3
	T	-3	1
		4	11

Difference of elements of column I

$$8 - (-3) = 11$$

Difference of elements in column II

$$-3 - 1 = -4$$

2 X 2 Games - Mixed Strategies Arithmetic Method - Example Problem

Taking the difference of two numbers in a row I (II), and put it under the row II (I), ignoring sign.

		B		
		H	T	
A	H	8	-3	4
	T	-3	1	11
		4	11	

Difference of elements of row I

$$8 - (-3) = 11$$

Difference of elements in row II

$$-3 - 1 = -4$$

2 X 2 Games - Mixed Strategies Arithmetic Method - Example Problem

Thus, for optimum gains, player A must use the strategy H with probability $\frac{4}{15}$ and strategy T with probability $\frac{11}{15}$

B

		H	T		
A	H	8	-3	4	$\frac{4}{4 + 11}$
	T	-3	1	11	$\frac{11}{4 + 11}$
		4	11		
		4	11		
		4 + 11	4 + 11		

Player B must use the strategy H with probability $\frac{4}{15}$ and strategy T with probability $\frac{11}{15}$.

2 X 2 Games - Mixed Strategies Arithmetic Method - Example Problem

Value of the game can be obtained as follows

		B		
		H	T	
A	H	8	-3	4
	T	-3	1	11
		4	11	

Using A's oddments

B plays H; value of the game

$$V = \frac{4 \times 8 - 3 \times 11}{4 + 11} = -\frac{1}{15}$$

B plays T; value of the game

$$V = \frac{-3 \times 4 + 1 \times 11}{4 + 11} = -\frac{1}{15}$$

2 X 2 Games - Mixed Strategies Arithmetic Method - Example Problem

Value of the game can be obtained as follows

		B		
		H	T	
A	H	8	-3	4
	T	-3	1	11
		4	11	

Using B's oddments

A plays H; value of the game

$$V = \frac{4 \times 8 - 3 \times 11}{4 + 11} = -\frac{1}{15}$$

A plays T; value of the game

$$V = \frac{-3 \times 4 + 1 \times 11}{4 + 11} = -\frac{1}{15}$$

Thus the solution of the game is $A\left(\frac{4}{15}, \frac{11}{15}\right)$, $B\left(\frac{4}{15}, \frac{11}{15}\right)$ and $V = -\frac{1}{15}$ is the gain to player A.

Dominance Property of Reducing the Size of the Game

- The size of the pay-off matrix of a game can be reduced sometimes by eliminating a course of action which is so inferior to another as never to be used. Such a course of action is said to be dominated by the other.
- It is very useful in reducing games of large size (order) to game of size 2 in at least one dimension, so that it can be solved easily.

Dominance Property of Reducing the Size of the Game

General rule

- If all the elements of a row, say k^{th} , are less than or equal to the corresponding elements of any other row, say, r^{th} , then k^{th} row is dominated by the r^{th} row.
- If all the elements of a column, say, k^{th} are greater than or equal to the corresponding elements of any other column, say, r^{th} , then, k^{th} column is dominated by r^{th} column.
- Omit dominated rows or columns.
- If some linear combination of some rows dominates i^{th} row, then i^{th} row will be deleted. Similar argument follows for columns also.

3 X 3 Game - Example Problem 1

Two players P and Q play a game. Each of them has to choose one of the three colors, white (W), black (B) and red(R) independently of the other. Thereafter the colors are compared. If both P and Q have chosen white (W,W), neither wins anything. If player P selects white and player Q black (W,B), player P loses ₹2 or player Q wins the same amount and so on. The complete payoff table is shown below. Find the optimum strategies for P and Q and the value of the game.

		Q		
		W	B	R
P	W	0	-2	7
	B	2	5	6
	R	3	-3	8

3 X 3 Game - Example Problem 1

Check whether saddle point exists or not.

		Q			
		W	B	R	Row min
P	W	0	-2	7	-2
	B	2	5	6	2
	R	3	-3	8	-3
Col max		3	5	8	

Since the Maximin and Minimax values are not equal, there is no saddle point and the players have to choose mixed strategies to maximize their gains.

3 X 3 Game - Example Problem

Dominance rule for columns is - Every value in the dominating column(s) must be less than or equal to the corresponding value of the dominated column.

		Q					Q	
		W	B	R			W	B
P	W	0	-2	7		W	0	-2
	B	2	5	6		B	2	5
	R	3	-3	8		R	3	-3

Strategy R is dominated by strategy W

3 X 3 Game - Example Problem 1

Dominance rule for rows is - Every value in the dominating row(s) must be greater than or equal to the corresponding value of the dominated row.

		Q	
		W	B
P	W	0	-2
	B	2	5
	R	3	-3



		Q	
		W	B
P	B	2	5
	R	3	-3



Can be solved
by arithmetic
method

Strategy W is dominated by strategy B

3 X 3 Game - Example Problem 2

Solve the game with the pay-off matrix given below.

		B		
		1	2	3
A	1	6	1	3
	2	0	9	7
	3	2	3	4

3 X 3 Game - Example Problem 2

Check whether the game has saddle point or not.

		B			Row min
		1	2	3	
A	1	6	1	3	1
	2	0	9	7	0
	3	2	3	4	2
	Col max	6	9	7	

No saddle point

3 X 3 Game - Example Problem 2

Trying to apply dominance rules to reduce the size of the matrix

		B		
		1	2	3
A	1	6	1	3
	2	0	9	7
	3	2	3	4

Trying to apply dominance rules to reduce the size of the matrix

There are no dominating rows or columns in the given pay-off matrix.

But when we consider the average of pay-offs of first two rows, they dominate the third row, hence it can be deleted.

$$\left(\frac{6+0}{2}, \frac{1+9}{2}, \frac{3+7}{2} \right) = (3, 5, 5) > (2, 3, 4)$$

3 X 3 Game - Example Problem 2

Therefore the reduced pay-off matrix becomes

		B					B		
		1	2	3			1	2	3
A	1	6	1	3	A	1	6	1	3
	2	0	9	7		2	0	9	7
	3	2	3	4		3			

→

Graphical Method(for $2 \times n$ or $m \times 2$ games)

- The graphical method is useful for the game where the pay-off matrix is of the size $2 \times n$ or $m \times 2$.
- Optimal strategies for both the players assign non zero probabilities to the same number of pure strategies. Therefore, if one player has only two strategies, the other will also use the same number of strategies.
- Hence this method is useful in finding out which of the two strategies can be used.

Games With Mixed Strategies

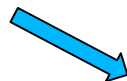
Graphical Solution Procedure

Graphical procedure is applicable, if one of the player has only two strategies.

Using dominance rule, player A has strategy III which is dominated by strategy II, hence can be eliminated.

	I	II	III
I	0	-2	2
II	5	4	-3
III	2	3	-4

(3x3) Game



	I	II	III
I	0	-2	2
II	5	4	-3

(2x3) Game

Games With Mixed Strategies

Graphical Solution Procedure

		q_1	q_2	q_3
		I	II	III
p_1	I	0	-2	2
p_2	II	5	4	-3

Since $p_1 + p_2 = 1$

Strategy	Expected payoff
I	$5 - 5p_1$
II	$4 - 6p_1$
III	$-3 + 5p_1$

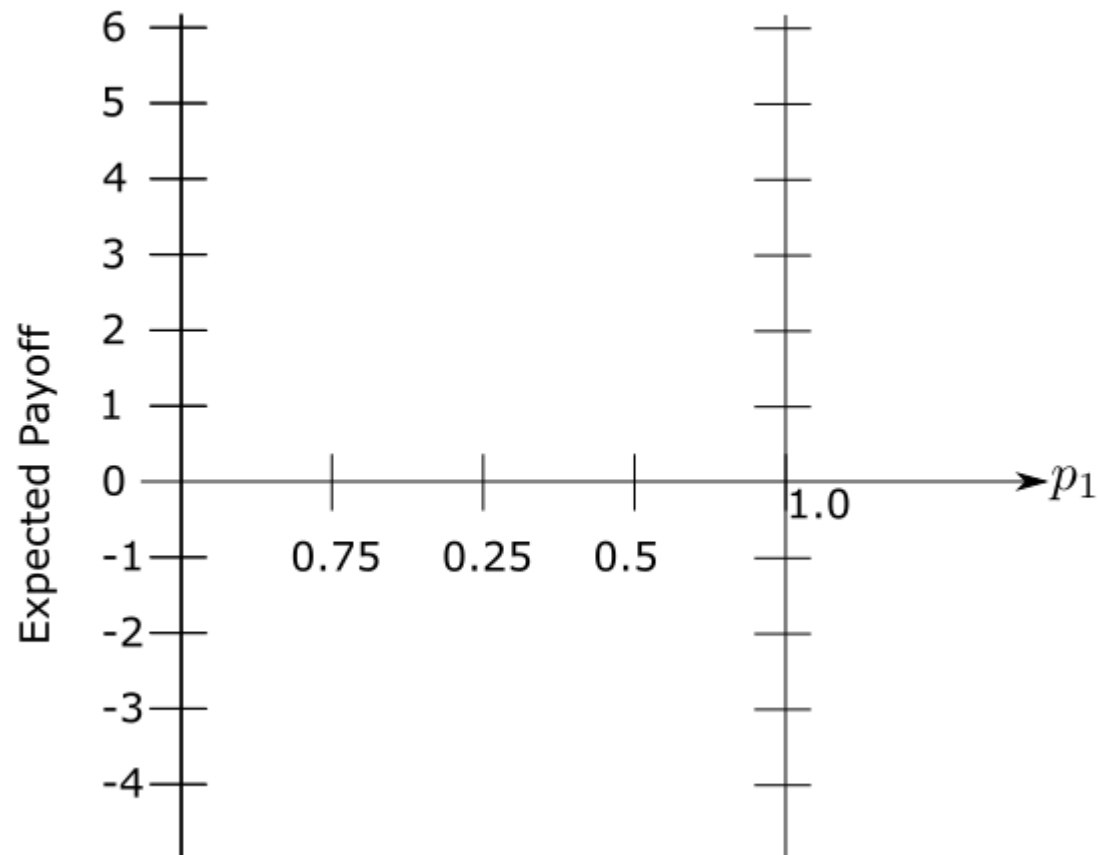
For each of the pure strategies available to player B, the expected payoff to player A will be

Strategy	Expected payoff
I	$0p_1 + 5p_2$
II	$-2p_1 + 4p_2$
III	$2p_1 - 3p_2$



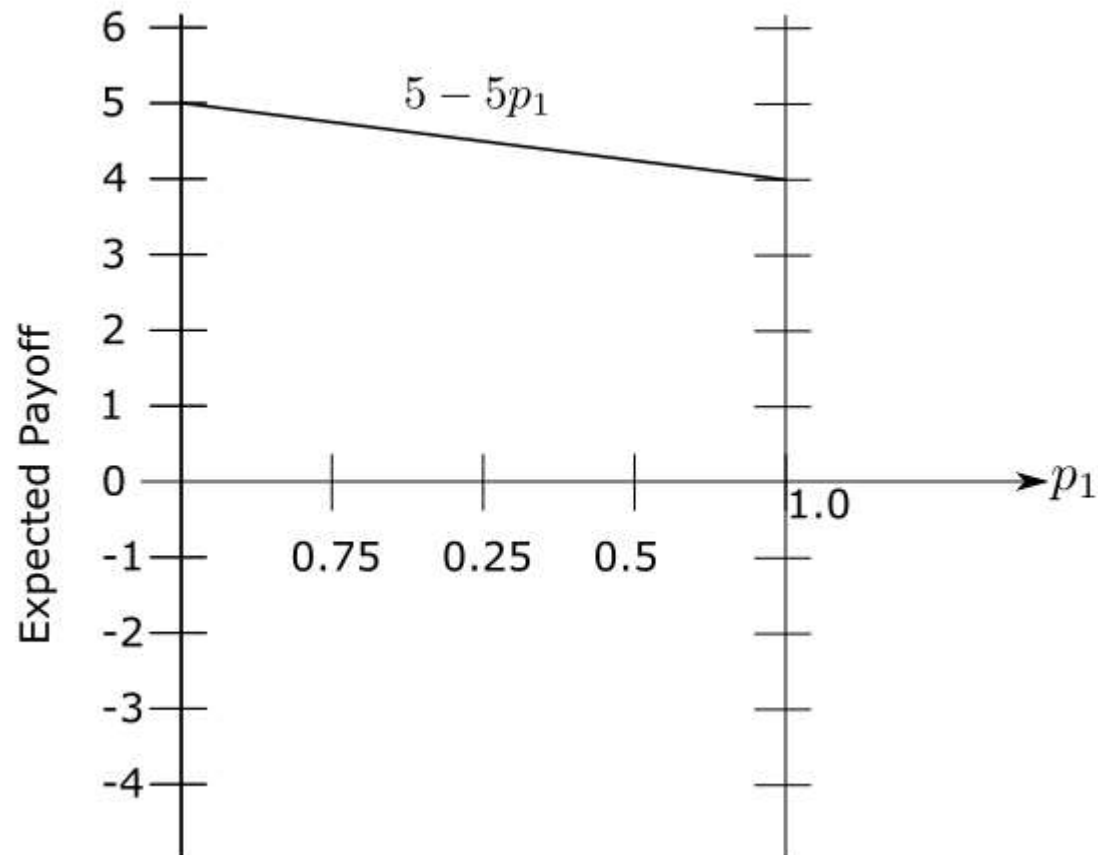
Games With Mixed Strategies

Graphical Solution Procedure



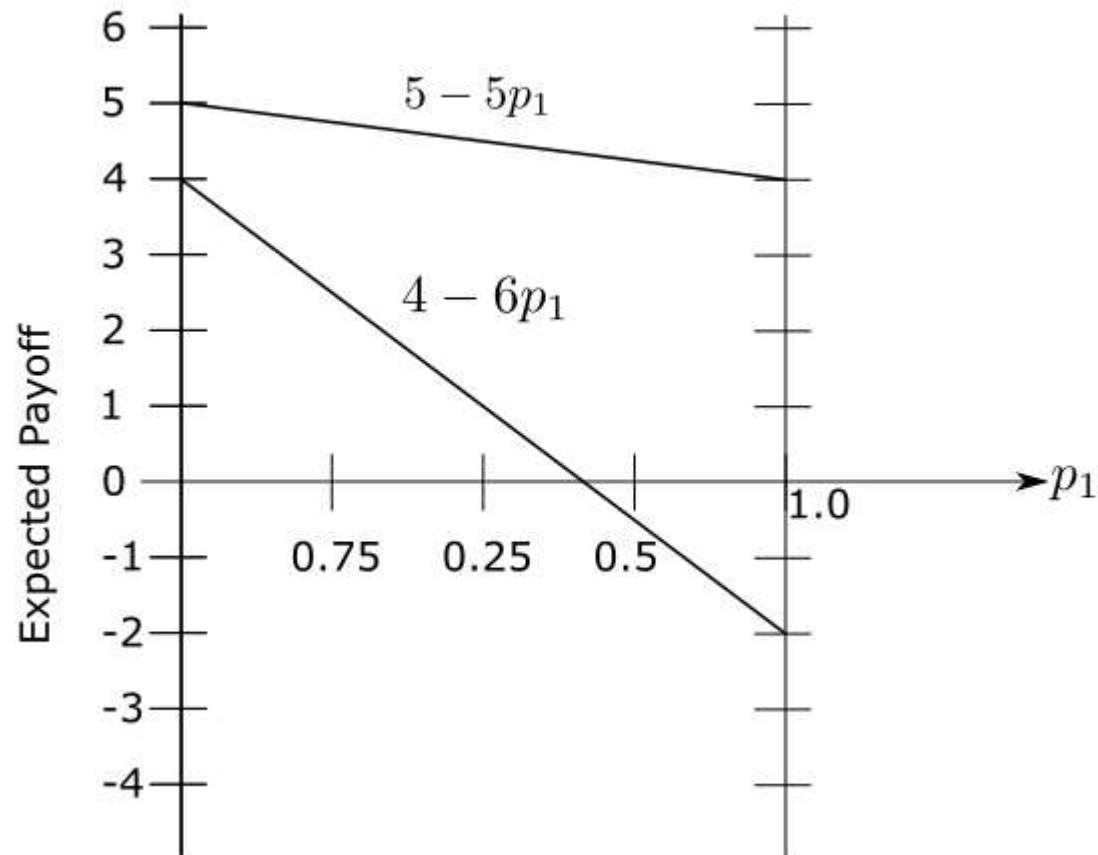
Games With Mixed Strategies

Graphical Solution Procedure



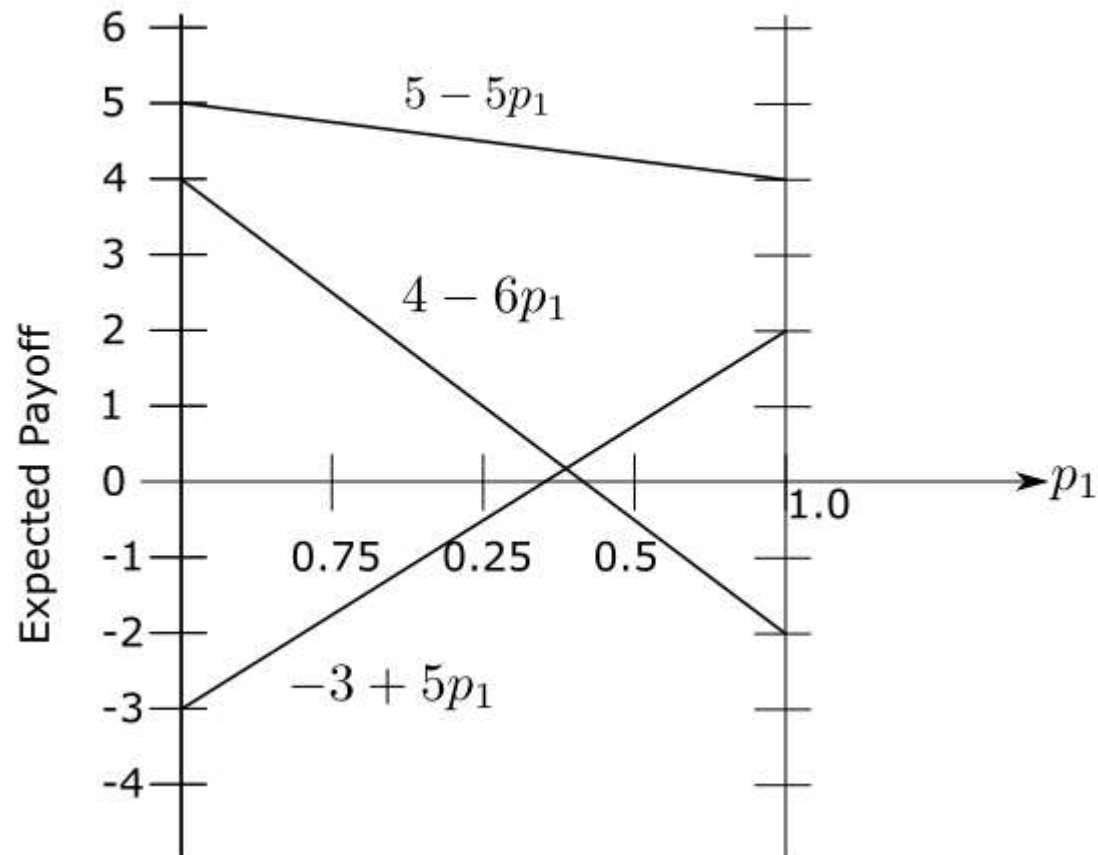
Games With Mixed Strategies

Graphical Solution Procedure



Games With Mixed Strategies

Graphical Solution Procedure



Games With Mixed Strategies Graphical Solution Procedure

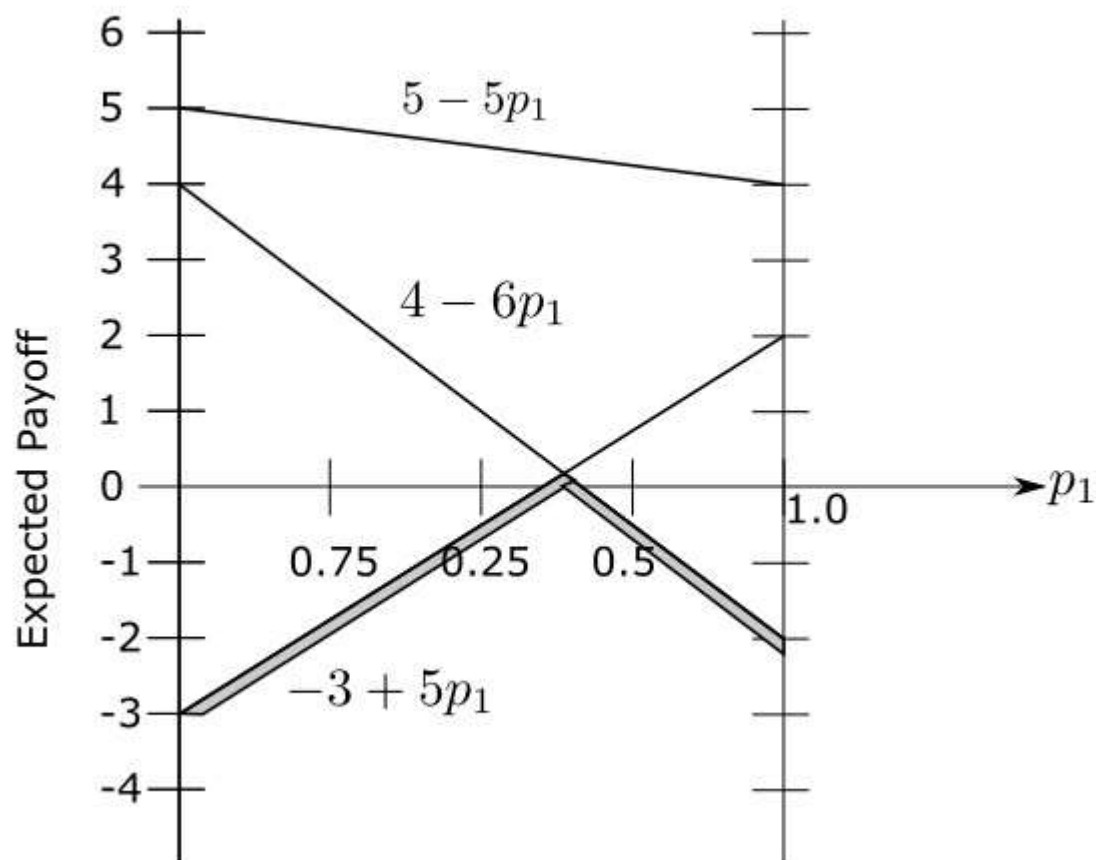
- For any value of p_1 and (q_1, q_2, q_3) , the expected payoff to player A will be appropriate weighted average of the corresponding points on these lines.
- Expected payoff to player A is

$$q_1(5 - 5p_1) + q_2(4 - 6p_1) + q_3(-3 + 5p_1)$$

- Since player B wants to minimize the expected payoff to player A, he minimizes this expected payoff by choosing the pure strategy that corresponds to bottom line for that p_1 .
- According to minimax criterion, player A should select the value of p_1 where the bottom line peak.

Games With Mixed Strategies

Graphical Solution Procedure



Games With Mixed Strategies Graphical Solution Procedure

The intersection of the two lines is obtained by setting

$$-3 + 5p_1 = 4 - 6p_1$$

Upon solving, we get

$$11p_1 = 7 \Rightarrow p_1 = \frac{7}{11}$$

and

$$p_2 = 1 - p_1 = 1 - \frac{7}{11} = \frac{4}{11}$$

Therefore value of the game is

$$u = -3 + 5p_1 = -3 + 5 \times \frac{7}{11} = \frac{2}{11}$$

Games With Mixed Strategies

Graphical Solution Procedure

		q_2	q_3
		II	III
p_1	I	-2	2
p_2	II	4	-3

Expected payoffs to player B, for each of the pure strategies of player A are

Strategy	Expected payoff
I	$-2q_2 + 2q_3$
II	$4q_2 - 3q_3$

$$-2q_2 + 2q_3 = 4q_2 - 3q_3$$

$$6q_2 = 5q_3$$

With optimal strategies

Games With Mixed Strategies Graphical Solution Procedure

Since $q_2 + q_3 = 1 \Rightarrow 5q_2 + 5q_3 = 5$

$$\begin{aligned} 5q_2 + 6q_2 &= 5 \\ 5 & \\ \Rightarrow q_2 &= \frac{5}{11} \end{aligned}$$

and

$$q_3 = 1 - q_2 = \frac{6}{11}$$

Therefore, the optimum strategies are,

Player A $\rightarrow (\frac{7}{11}, \frac{4}{11}, 0)$

Player B $\rightarrow (0, \frac{5}{11}, \frac{6}{11})$

Check

Value of Game

$$\begin{aligned} &4q_2 - 3q_3 \\ &4\left(\frac{5}{11}\right) - 3\left(\frac{6}{11}\right) \\ &\frac{20}{11} - \frac{18}{11} \\ &\frac{2}{11} \end{aligned}$$

Player A gains $\frac{2}{11}$

Graphical Method(for 2 x n or m x 2 games)

Obtain the optimal strategies for both persons and the value of the game for zero sum two person game whose pay off matrix is as follows.

		B	
		B_1	B_2
A	A_1	1	-3
	A_2	3	5
	A_3	-1	6
	A_4	4	1
	A_5	2	2
	A_6	-5	0

Graphical Method(for $2 \times n$ or $m \times 2$ games)

The game does not have a saddle point. Let the probability of player B playing B_1 and B_2 in the mixed strategy is denoted by q_1 and q_2 respectively. Similarly let p_1, p_2, p_3, p_4, p_5 and p_6 be the probabilities with which A selects his strategies A_1, A_2, A_3, A_4, A_5 and A_6 respectively.

Graphical Method(for 2 x n or m x 2 games)

			B	
			q_1	q_2
			B_1	B_2
A	p_1	A_1	1	-3
	p_2	A_2	3	5
	p_3	A_3	-1	6
	p_4	A_4	4	1
	p_5	A_5	2	2
	p_6	A_6	-5	0

Graphical Method(for 2 x n or m x 2 games)

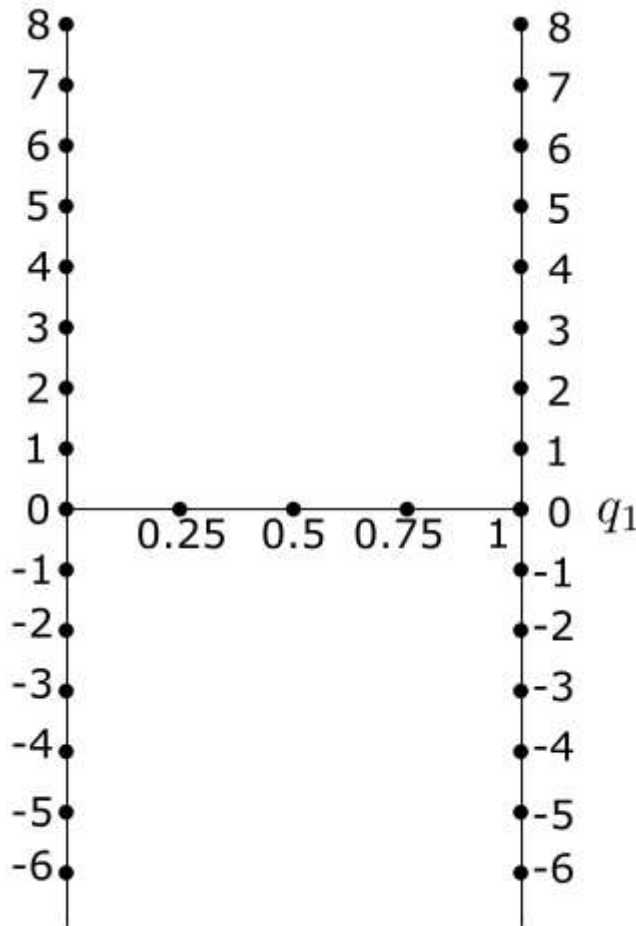
The expected pay off to player B for each strategy of player A is

		B	
		q_1	q_2
		B_1	B_2
A	p_1 A_1	1	-3
	p_2 A_2	3	5
	p_3 A_3	-1	6
	p_4 A_4	4	1
	p_5 A_5	2	2
	p_6 A_6	-5	0

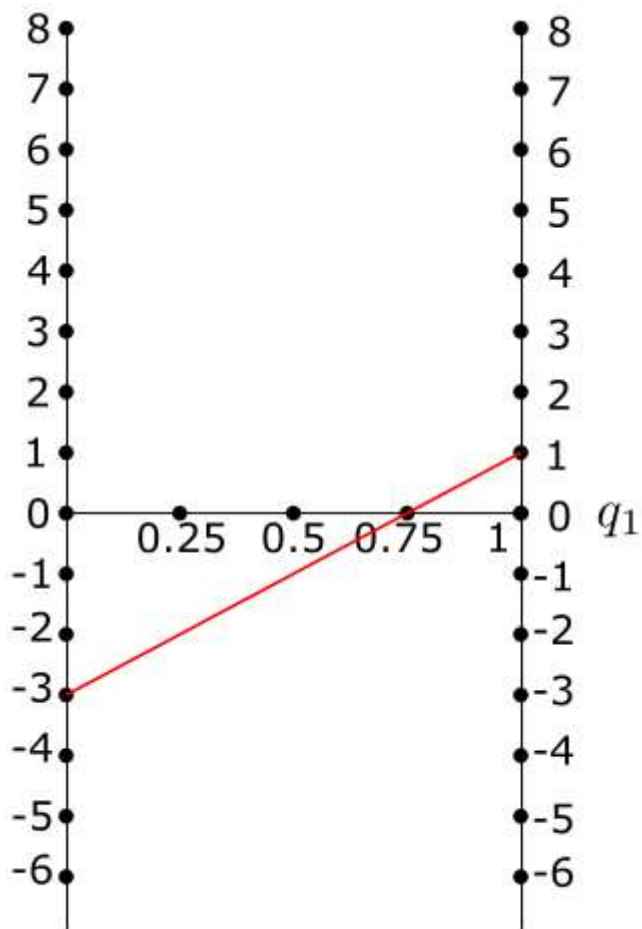
A's pure strategies	B's expected pay-offs
A_1	$q_1 - 3q_2$
A_2	$3q_1 + 5q_2$
A_3	$-q_1 + 6q_2$
A_4	$4q_1 + q_2$
A_5	$2q_1 + 2q_2$
A_6	$-5q_1 + 0q_2$

Graphical Method(for 2 x n or m x 2 games)

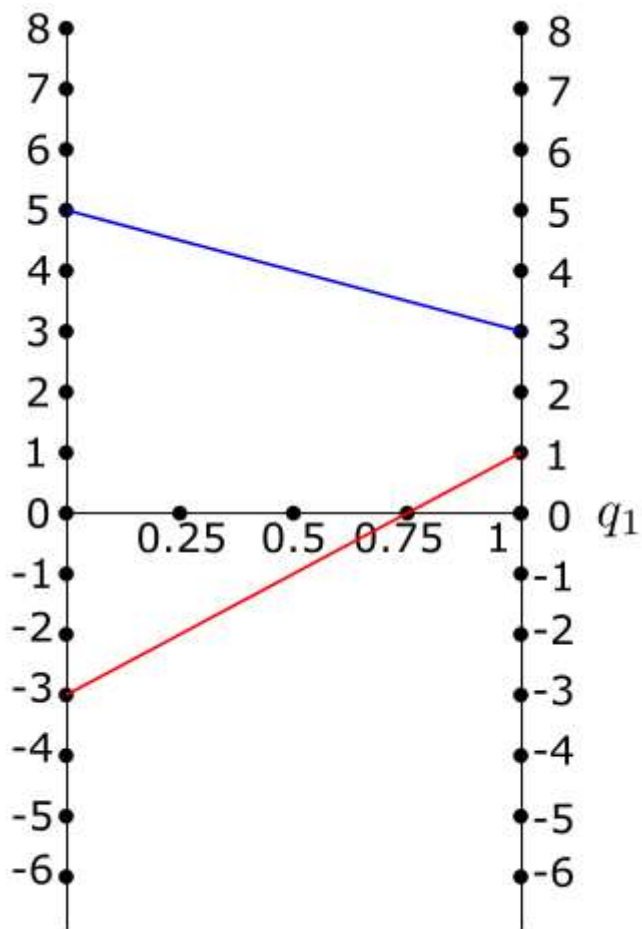
The expected pay off equations are then plotted as functions of q_i



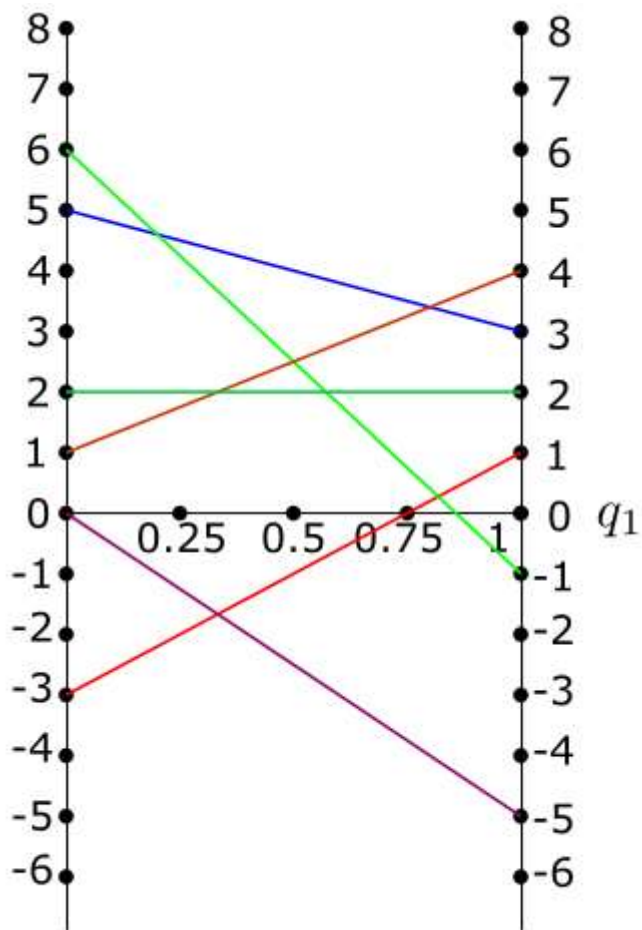
Graphical Method(for 2 x n or m x 2 games)



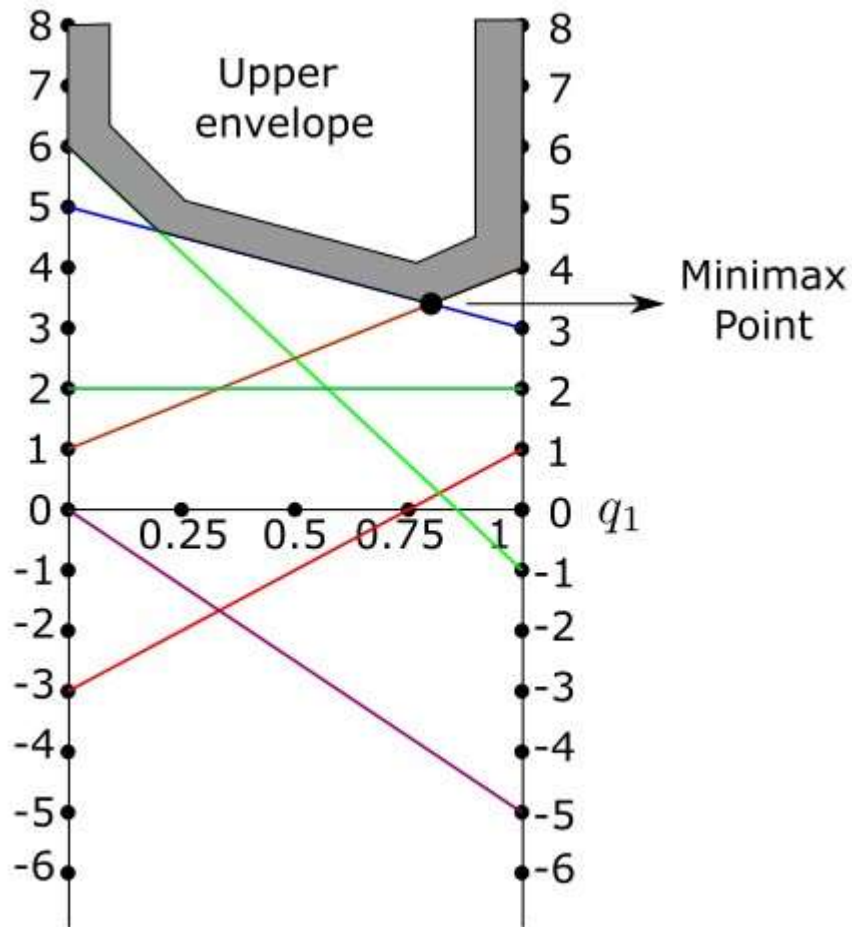
Graphical Method(for 2 x n or m x 2 games)



Graphical Method(for 2 x n or m x 2 games)



Graphical Method(for 2 x n or m x 2 games)



Since player B wishes to minimize his maximum expected pay-offs, we consider the minimax point on the upper envelope of B's expected pay-off equations. Hence, the given pay-off matrix of the game is reduced to

		B	
		B_1	B_2
A	A_2	3	5
	A_4	4	1

Graphical Method (for 2 x n or m x 2 games)

Using arithmetic method to solve the mixed problem

		B		
		B_1	B_2	
A	A_2	3	5	3
	A_4	4	1	2
		4	1	
		$\frac{4}{4+1} = 0.8$	$\frac{1}{4+1} = 0.2$	

$$\frac{3}{3+2} = 0.6$$

$$\frac{2}{2+3} = 0.4$$

Graphical Method (for 2 x n or m x 2 games)

Using B's oddments

Value of the game is

$$V = \frac{3 \times 3 + 4 \times 2}{3 + 2} = \frac{17}{5}$$

And the optimal strategies are

For player A

$$\begin{bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ 0 & 0.6 & 0 & 0.4 & 0 & 0 \end{bmatrix}$$

For player B

$$\begin{bmatrix} B_1 & B_2 \\ 0.8 & 0.2 \end{bmatrix}$$

Game without Saddle Point Example 3

Any game with mixed strategies can be solved by transforming the problem into a linear programming problem.

Consider the following game

		B		
A		0	2	2
		3	-1	3
		4	4	-2

Game without Saddle Point Example 3

Since the game does not have a saddle point, then player should use mixed strategies. Let p_1, p_2 and p_3 be the probabilities with which player A selects his 1,2 and 3rd strategies respectively. q_1, q_2 & q_3 be the probabilities with which player B selects his 1,2 and 3rd strategies respectively.

	q_1	q_2	q_3
p_1	0	2	2
p_2	3	-1	3
p_3	4	4	-2

Game without Saddle Point Example 3

For each pure strategy of player B, player A gains are

$$3p_2 + 4p_3$$

$$2p_1 - p_2 + 4p_3$$

$$2p_1 + 3p_2 - 2p_3$$

	q_1	q_2	q_3
p_1	0	2	2
p_2	3	-1	3
p_3	4	4	-2

Taking u as the minimum gain that player B allows player A to have, we can write

$$3p_2 + 4p_3 \geq u$$

$$2p_1 - p_2 + 4p_3 \geq u$$

$$2p_1 + 3p_2 - 2p_3 \geq u$$

Game without Saddle Point Example 3

Now player A's problem can be expressed as

Maximize minimum gain $u \Rightarrow$ Maximize u

satisfying

$$u - 3p_2 - 4p_3 \leq 0$$

$$u - 2p_1 + p_2 - 4p_3 \leq 0$$

$$u - 2p_1 - 3p_2 + 2p_3 \leq 0$$

$$p_1 + p_2 + p_3 = 1$$

and

$$p_1, p_2, p_3 \geq 0$$

u is unrestricted

Game without Saddle Point Example 3

For each pure strategy of player A, player B losses are

$$2q_2 + 2q_3$$

$$3q_1 - q_2 + 3q_3$$

$$4q_1 + 4q_2 - 2q_3$$

	q_1	q_2	q_3
p_1	0	2	2
p_2	3	-1	3
p_3	4	4	-2

Taking v as the maximum loss that player A inflict on player B, we can write

$$2q_2 + 2q_3 \leq v$$

$$3q_1 - q_2 + 3q_3 \leq v$$

$$4q_1 + 4q_2 - 2q_3 \leq v$$

Game without Saddle Point Example 3

Now player B's problem can be expressed as

Minimize maximum loss $v \Rightarrow$ Minimize v

satisfying

$$v - 2q_2 - 2q_3 \geq 0$$

$$v - 3q_1 + q_2 - 3q_3 \geq 0$$

$$v - 4q_1 - 4q_2 + 2q_3 \geq 0$$

$$q_1 + q_2 + q_3 = 1$$

and

$$q_1, q_2, q_3 \geq 0$$

v is unrestricted

Game without Saddle Point Example 3

Primal

Maximize u

Subjected to

$$u - 3p_2 - 4p_3 \leq 0$$

$$u - 2p_1 + p_2 - 4p_3 \leq 0$$

$$u - 2p_1 - 3p_2 + 2p_3 \leq 0$$

$$p_1 + p_2 + p_3 = 1$$

and

$$p_1, p_2, p_3 \geq 0$$

u is unrestricted

Dual

Minimize w

Subjected to

$$y_1 + y_2 + y_3 = 1$$

$$-2y_2 - 2y_3 + w \geq 0$$

$$-3y_1 + y_2 - 3y_3 + w \geq 0$$

$$-4y_1 - 4y_2 + 2y_3 + w \geq 0$$

and

$$y_1, y_2, y_3 \geq 0$$

w is unrestricted

Game without Saddle Point Example 3

- Game of player B is the dual of the game of player A. Hence one has to solve for game of player A only and can get the solution of game of other player B using duality principle.
- Simplex method can be used to solve the linear programming problem formulated for the game of the either player, which gives the solution of the game of other player also.

Suggested Reading List

1. S D Sharma, Operations Research - Theory, Methods and Applications, 5th edition, Kedarnath - Ramnath & Co.
2. H A Taha, Operations Research - An Introduction, 8th edition, Pearson Prentice Hall
3. A M Natarajan, P Balasubramanei, A Tamilarasi, Operations Research, 1st edition, Pearson.
4. J K Sharma, Operations Research - Problems and Solutions, 3rd edition, MacMillan Publishers India.

For more course related information, please visit

<https://sites.google.com/view/operationsresearchgiet/home>

Please access the google class room to post queries, download course materials, attempt quizzes and assignments.

OPERATIONS RESEARCH

UNIT 4

Lecture 1



Presented By:

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Assistant Professor

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Lecture Details:

Introduction to Inventory Theory, Basic EOQ Model

Operations Research/ME, 5 Semester.

LEARNING OUTCOMES

Upon completion of this lecture, student should be able to

- Explain the importance of inventory control
- Describe various costs associated with inventory control
- List various types of inventories
- Determine optimal inventory policy for simple inventory problems with known demand

Introduction

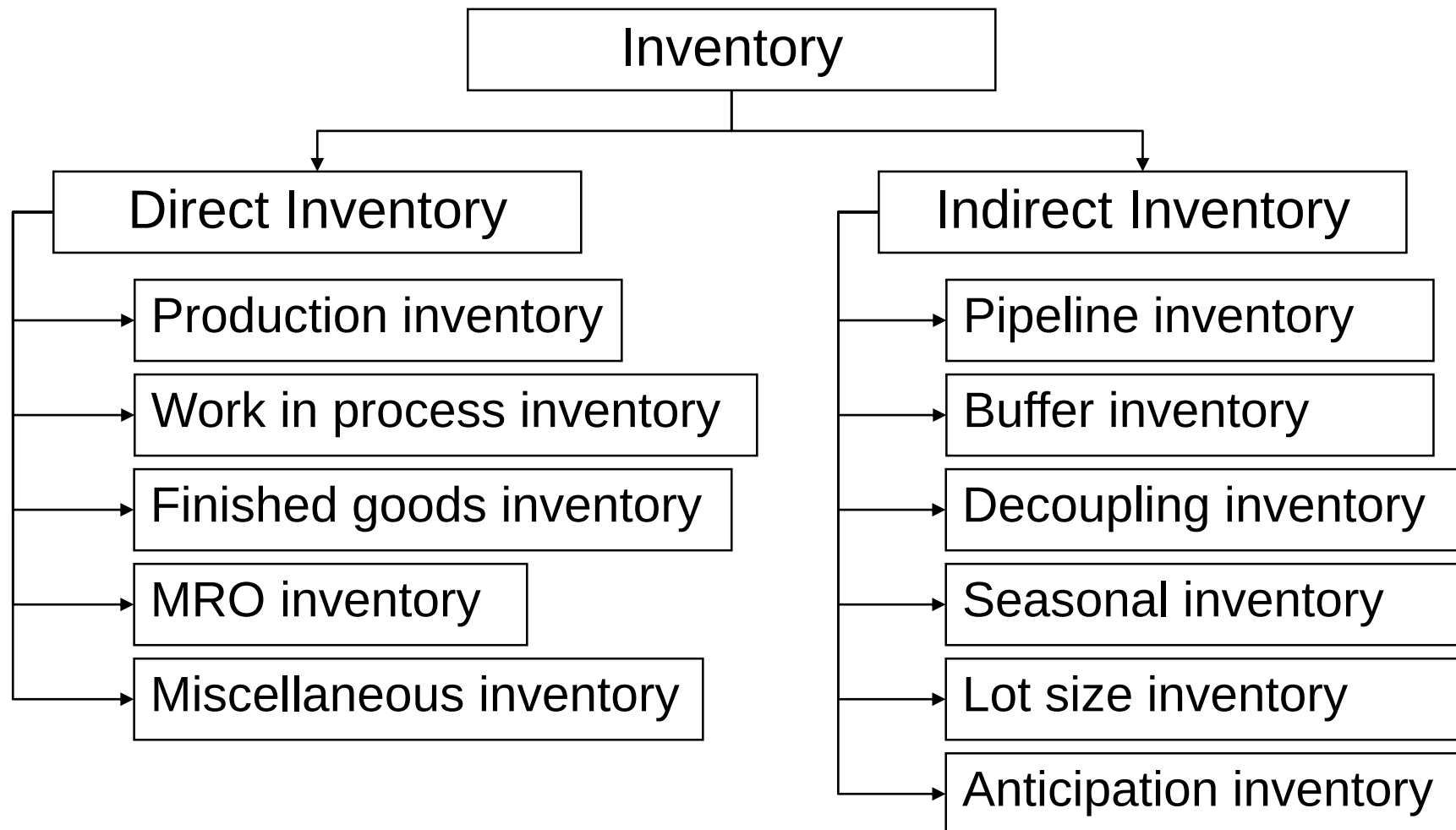
- ☞ Inventory is an idle resource of an enterprise which is kept for future use.
- ☞ The resource can be men, machine, material or money.
 - ❖ STOCK – Material kept as idle resource for future use.
- ☞ Inventory model deals with determining the level of inventory of an item that an enterprise must maintain to ensure smooth operation.

WHEN TO BUY? & HOW MUCH TO BUY?

Necessity for Maintaining Inventory

- ☞ It helps in smooth and efficient running of an enterprise by decoupling the production from customers and vendors.
- ☞ It helps in providing timely deliveries thereby increasing customer satisfaction and goodwill of the enterprise.
- ☞ It helps the enterprise to take the benefits of price discount through bulk purchase.
- ☞ It reduces the production cost through batch production which is long and uninterrupted.
- ☞ It helps in efficient and effective utilization of human resource.
- ☞ It acts as buffer stock against late shipments and shop rejections.

Classification of Inventories



Inventory Costs

- ☞ The objective of the inventory model is to minimize the *total inventory cost*.

$$\left(\begin{array}{c} \text{Total} \\ \text{inventory} \\ \text{cost} \end{array} \right) = \left(\begin{array}{c} \text{Purchase} \\ \text{cost} \end{array} \right) + \left(\begin{array}{c} \text{Holding} \\ \text{cost} \end{array} \right) + \left(\begin{array}{c} \text{Ordering} \\ \text{cost} \end{array} \right) + \left(\begin{array}{c} \text{Shortage} \\ \text{cost} \end{array} \right)$$

- ☞ Purchase cost can be variable or fixed cost with respect to order quantity, but holding, ordering and shortage costs are variable costs and depend on the order quantity.

Inventory Costs

☞ Inventory Carrying Cost or Holding Cost

- ❖ Cost of capital tied up in inventory
- ❖ Cost of storage space
- ❖ Depreciation and deterioration cost
- ❖ Pilferage cost
- ❖ Obsolescence cost
- ❖ Handling cost
- ❖ Record keeping and administrative costs
- ❖ Taxes and insurance

☞ It is expressed as *% total inventory/unit time* or *₹/unit/unit time*

Inventory Costs

☞ Ordering Costs or Setup Costs

❖ These are the fixed costs associated with placing an order or setting up a machinery before starting production.

- Cost of requisition
- Cost of follow up
- Cost of receiving goods
- Cost of quality control
- Cost of mailing, telephone, internet
- Salary of accounting and auditing personnel

☞ It is expressed as ₹/order or ₹/setup

Inventory Costs

☞ Shortage Costs or Stock Out Costs

- ❖ These are the costs incurred with either a delay in meeting the demand or inability to meet the demand at all.
- ❖ If the unfilled demand can be filled at a later stage through backorder, it is expressed as ₹/unit back order/unit time
- ❖ If the unfilled demand cannot be filled at a later stage, it is expressed as ₹/unit short

☞ What is the result?

Cancelled orders, Lost sales, Customer good will

Demand

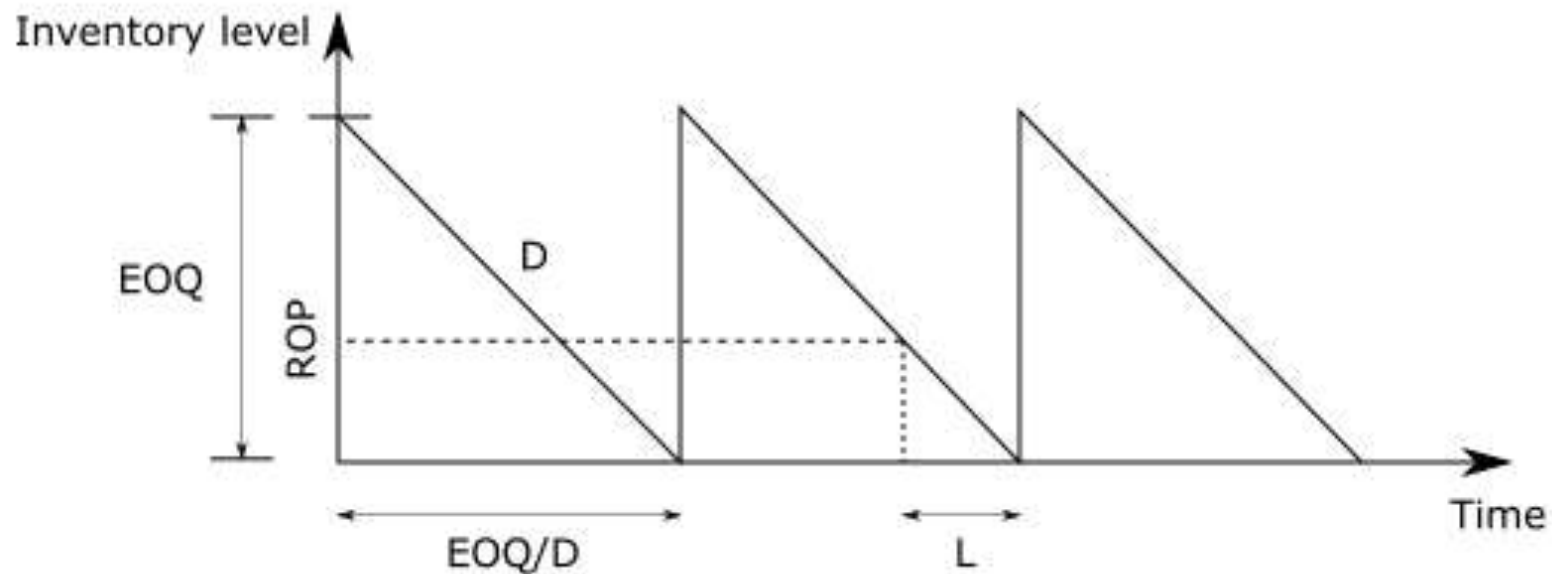
- ☞ The demand for a product in inventory is the number of units that will need to be withdrawn from inventory for some use (sales/production) during a specific period.
- ☞ If the demand in future periods can be forecast with considerable precision, it is considered to be a known demand and is said to be *deterministic*
- ☞ If the demand in future period cannot be forecast very well, it is considered to be a random variable and is said to be *probabilistic or stochastic demand*

Classification of Inventory Models

- ☞ Inventory models are classified as *deterministic* or *stochastic* models according to whether the demand for a period is known or is a random variable having a known probability distribution.
- ☞ Inventory models are also classified as *continuous review* or *periodic review* models according to whether the current inventory level is monitored continuously or periodically.
 - ❖ In continuous review model, an order is placed as soon as the stock level falls down to the prescribed **reorder point**, while in periodic review model, orders are placed only at fixed intervals even when the inventory level dips below reorder point.

Deterministic Continuous Review Model

(known demand, infinite replenishment rate, no shortage allowed, constant lead time)



D : - Demand Rate,

EOQ : - Economic Order Quantity

L : - Lead time

ROP : - Reorder point

Deterministic Continuous Review Model

(known demand, infinite replenishment rate, no shortage allowed, constant lead time)

Economic order quantity (units/order)

$$EOQ = \sqrt{\frac{2DC_0}{C_h}}$$

Here

D = Demand rate (units/unit time)

C_0 = Ordering cost (₹/order)

C_h = Carrying cost (₹/unit/unit time)

Cycle length (time)

$$t_{cycle} = \frac{EOQ}{D}$$

Deterministic Continuous Review Model

(known demand, infinite replenishment rate, no shortage allowed, constant lead time)

Reorder point (units)

$$ROP = L_e \times D$$

Where

$$\begin{aligned} L_e &= \text{Effective lead time} \\ &= L, \text{ if } L < t_{cycle} \\ &= L - n t_{cycle}, \text{ if } L > t_{cycle} \end{aligned}$$

$$n = \left(\text{largest integer} \leq \frac{L}{t_{cycle}} \right)$$

Deterministic Continuous Review Model

(known demand, infinite replenishment rate, no shortage allowed, constant lead time)

Optimum total variable inventory cost (₹/unit time)

$$TVCU = \frac{D}{EOQ} \times C_0 + \frac{EOQ}{2} \times C_h$$

Optimum total variable inventory cost (₹/unit time)

$$TCU = D \times C + \frac{D}{EOQ} \times C_0 + \frac{EOQ}{2} \times C_h$$

Deterministic Continuous Review Model

(known demand, infinite replenishment rate, no shortage allowed, constant lead time)



Example 1

A company uses an item at an approximate customer rate of 100 per day. Each unit costs ₹10 per item and the company personnel estimate that it costs ₹100 to place an order, and that the carrying cost of inventory is ₹ 0.02 per day. The lead time between placing and receiving an order is 5 days. Determine the optimal inventory policy for ordering the items. If the lead time is 12 days, what would be the optimal policy?

Deterministic Continuous Review Model

(known demand, infinite replenishment rate, no shortage allowed, constant lead time)

The economic order quantity is,

$$EOQ = \sqrt{\frac{2DC_0}{C_h}} = \sqrt{\frac{2 \times 100 \times 100}{0.02}} = 1000$$

Cycle length is,

$$t_{opt} = \frac{EOQ}{D} = \frac{1000}{100} = 10 \text{ days}$$

Reorder point

$$ROP = L_e \times D = LD \because L < t_{opt}$$

$$ROP = 5 \times 100 = 500 \text{ units}$$

The inventory policy is

“Order 1000 items whenever the inventory level drops to 500 units”

Deterministic Continuous Review Model

(known demand, infinite replenishment rate, no shortage allowed, constant lead time)

The daily inventory cost associated with the proposed inventory policy is

$$TVC = \frac{D}{EOQ} \times C_0 + \frac{EOQ}{2} \times C_h$$

$$TVC = \frac{100}{1000} \times 100 + \frac{1000}{2} \times 0.02 = ₹20/day$$

Deterministic Continuous Review Model

(known demand, infinite replenishment rate, no shortage allowed, constant lead time)



Example 2

The production department for a company requires 3,600 kg of raw material for manufacturing a particular item per year. It has been estimated that the cost of placing an order is ₹36 and the cost of carrying inventory is 25 per cent of the investment in the inventories. The price is ₹10 per kg. The purchase manager wishes to determine an ordering policy for the raw material.

Deterministic Continuous Review Model

(known demand, infinite replenishment rate, no shortage allowed, constant lead time)

➡ Economic order quantity,

$$EOQ = \sqrt{\frac{2DC_0}{C_h}} = \sqrt{\frac{2 \times 3600 \times 36}{0.25 \times 10}} = 322 \text{ units}$$

➡ Cycle length,

$$t_{opt} = \frac{EOQ}{D} = \frac{322}{3600} = 0.089 \text{ years}$$

➡ Minimum total inventory cost is,

$$TC = D \times C + \frac{D}{EOQ} \times C_0 + \frac{EOQ}{2} \times C_h$$
$$TC = 3600 \times 10 + \frac{3600}{322} \times 36 + \frac{322}{2} \times 2.5 = ₹36805$$

Bibliography

1. S D Sharma, Operations Research – Theory, Methods and Applications, 15th edition, Kedar Nath Ram Nath
2. H A Taha, Operations Research – An Introduction, 8th edition, Pearson Prentice Hall
3. F S Hillier, G J Liberman, Introduction to Operations Research, 9th edition, McGraw-Hill publication
4. Prem Kumar Gupta, Hira D S, Operations Research, 7th edition, S Chand Company PVT Ltd

OPERATIONS RESEARCH

UNIT 4

Lecture 2



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Lecture Details:

Basic Inventory Models – Finite Replenishment & Shortages

Operations Research/ME, 5 Semester.

LEARNING OUTCOMES

Upon completion of this lecture, student should be able to

- Determine optimal inventory policy for simple inventory problems with known demand

Deterministic Continuous Review Model

(known demand, finite replenishment rate, no shortage allowed)

Economic Batch Quantity

$$EBQ = \sqrt{\frac{d}{d - D}} \sqrt{\frac{2DC_0}{C_h}}$$

Here

d = Production rate

D = Demand rate

Optimum Cycle Time

$$t_{opt} = \frac{EBQ}{D}$$

Deterministic Continuous Review Model

(known demand, finite replenishment rate, no shortage allowed)

Maximum Inventory

$$Q_{max} = \frac{d - D}{d} \times EBQ$$

Minimum Inventory Cost

$$TC = \sqrt{\frac{d - D}{d}} \sqrt{2DC_0C_h}$$

Manufacturing Time

$$t_{mfg} = \frac{EBQ}{d}$$

Deterministic Continuous Review Model

(known demand, finite replenishment rate, no shortage allowed)



Example 1

A company has to supply 10000 bearings per day to an automobile manufacturer. The production rate of the company is 25000 bearings per day. The cost of holding a bearing in stock for one year is 2 paisa and the set up cost of a production run is ₹18. Determine the optimum inventory policy.

Deterministic Continuous Review Model

(known demand, finite replenishment rate, no shortage allowed)

Economic Batch Quantity

$$EBQ = \sqrt{\frac{d}{d - D}} \sqrt{\frac{2DC_0}{C_h}}$$

$$EBQ = \sqrt{\frac{25000}{25000 - 10000}} \sqrt{\frac{2 \times 10000 \times 18}{0.02/360}}$$

$$EBQ = 104446.6 \text{ bearings}$$

Deterministic Continuous Review Model

(known demand, finite replenishment rate, no shortage allowed)

Cycle length

$$t_{opt} = \frac{EBQ}{D} = \frac{104446.6}{10000} = 10.4 \text{ days}$$

Length of the production run

$$t_{mfg} = \frac{EBQ}{d} = \frac{104446.6}{25000} = 4 \text{ days}$$

“production cycle starts at an intervals of 10.4 days and production continues for 4 days and in each batch 104446.6 bearings are produced”

Deterministic Continuous Review Model

(known demand, finite replenishment rate, no shortage allowed)

Optimum inventory cost

$$TC = \sqrt{\frac{d - D}{d}} \sqrt{2DC_0C_h}$$

$$TC = \sqrt{\frac{25000 - 10000}{25000}} \sqrt{2 \times 10000 \times 18 \times 0.000055}$$

$$TC = ₹ 3.466 \text{ per day}$$

Deterministic Continuous Review Model

(known demand, infinite replenishment rate, shortages allowed, no lead time)

Economic Order Quantity

$$EOQ = \sqrt{\frac{C_s + C_h}{C_s}} \sqrt{\frac{2DC_0}{C_h}}$$

Here

C_s = Shortage cost (₹/unit/unit time)

Optimum Cycle Time

$$t_{opt} = \frac{EOQ}{D}$$

Deterministic Continuous Review Model

(known demand, infinite replenishment rate, shortages allowed, no lead time)

Optimum Inventory Cost

$$TC = \sqrt{\frac{C_s}{C_s + C_h}} \sqrt{2DC_0C_h}$$

Optimum Shortage Quantity

$$S = \sqrt{\frac{2DC_0}{C_s}} \sqrt{\frac{C_h}{C_s + C_h}}$$

Deterministic Continuous Review Model

(known demand, infinite replenishment rate, shortages allowed, no lead time)



Example 1

A commodity is to be supplied at a constant rate of 200 units per day. Supplies of any amounts can be had at any required time, but each ordering costs ₹50.00. Cost of holding the commodity in inventory is ₹2.00 per unit per day while delay in the supply of the items induces a penalty of ₹10.00 per unit per delay of 1 day. Determine the optimal inventory policy.

Deterministic Continuous Review Model

(known demand, infinite replenishment rate, shortages allowed, no lead time)

Economic order quantity

$$EOQ = \sqrt{\frac{C_s + C_h}{C_s}} \sqrt{\frac{2DC_0}{C_h}}$$
$$EOQ = \sqrt{\frac{10 + 2}{10}} \sqrt{\frac{2 \times 200 \times 50}{2}} \cong 110 \text{ units}$$

Optimum cycle length

$$t_{opt} = \frac{EOQ}{D} = \frac{110}{200} = 0.55 \text{ days}$$

Deterministic Continuous Review Model

(known demand, infinite replenishment rate, shortages allowed, no lead time)

Optimum shortage allowed

$$S = \sqrt{\frac{2DC_0}{C_s}} \sqrt{\frac{C_h}{C_s + C_h}}$$
$$S = \sqrt{\frac{2 \times 200 \times 50}{10}} \sqrt{\frac{2}{10 + 2}} \cong 18 \text{ units}$$

Optimum inventory cost

$$TC = \sqrt{\frac{C_s}{C_s + C_h}} \sqrt{2DC_0C_h} = ₹182.58/\text{day}$$

Deterministic Continuous Review Model

(known demand, finite replenishment rate, shortages allowed, no lead time)

Economic Batch Quantity

$$EBQ = \sqrt{\frac{d}{d-D}} \sqrt{\frac{C_s + C_h}{C_s}} \sqrt{\frac{2DC_0}{C_h}}$$

Optimum shortage allowed

$$S = \sqrt{\frac{2DC_0}{C_s}} \sqrt{\frac{C_h}{C_s + C_h}} \sqrt{\frac{d-D}{d}}$$

Optimum Total Inventory Cost

$$TC = \sqrt{\frac{d-D}{d}} \sqrt{\frac{C_s}{C_s + C_h}} \sqrt{2DC_0C_h}$$

Deterministic Continuous Review Model

(known demand, finite replenishment rate, shortages allowed,
no lead time)



Example 1

The demand for an item in a company is 18,000 units per year, and the company can produce the item at a rate of 3,000 per month. The cost of one set-up is ₹500 and the holding cost of 1 unit per month is 15 paise. The shortage cost of one unit is ₹20 per month. Determine (i) optimum production batch quantity and the number of shortages; (ii) optimum cycle time and production time; (iii) maximum inventory level in the cycle; and (iv) total associated cost per year if the cost of the item is ₹20 per unit.

Deterministic Continuous Review Model

(known demand, finite replenishment rate, shortages allowed,
no lead time)

Economic Batch Quantity

$$EBQ = \sqrt{\frac{d}{d - D}} \sqrt{\frac{C_s}{C_s + C_h}} \sqrt{\frac{2DC_0}{C_h}}$$

$$EBQ = \sqrt{\frac{3000}{3000 - 1500}} \sqrt{\frac{20}{20 + 0.15}} \sqrt{\frac{2 \times 1500 \times 500}{0.15}}$$

$$EBQ = 4453.6 \text{ units}$$

Deterministic Continuous Review Model

(known demand, finite replenishment rate, shortages allowed,
no lead time)

Optimum Shortage Quantity

$$S = \sqrt{\frac{2DC_0}{C_s}} \sqrt{\frac{C_h}{C_s + C_h}} \sqrt{\frac{d - D}{d}}$$

$$S = \sqrt{\frac{2 \times 1500 \times 500}{20}} \sqrt{\frac{0.15}{20 + 0.15}} \sqrt{\frac{3000 - 1500}{3000}}$$

$$S \cong 17 \text{ units}$$

Deterministic Continuous Review Model

(known demand, finite replenishment rate, shortages allowed,
no lead time)

Optimum Production Run

$$t_{mfg} = \frac{EBQ}{d}$$

$$t_{mfg} = \frac{4453.6}{3000} \cong 1.5 \text{ months}$$

Optimum order cycle

$$t_{cycle} = \frac{EBQ}{D} = \frac{4453.6}{1500} \cong 3 \text{ months}$$

Deterministic Continuous Review Model

(known demand, finite replenishment rate, shortages allowed,
no lead time)

Total Inventory Cost

$$TC = \sqrt{\frac{d - D}{d}} \sqrt{\frac{C_s}{C_s + C_h}} \sqrt{2DC_0C_h}$$

$$TC = \sqrt{\frac{3000 - 1500}{3000}} \sqrt{\frac{20}{20 + 0.15}} \sqrt{2 \times 1500 \times 500 \times 0.15}$$

$$TC = ₹334/month$$

Bibliography

1. S D Sharma, Operations Research – Theory, Methods and Applications, 15th edition, Kedar Nath Ram Nath
2. H A Taha, Operations Research – An Introduction, 8th edition, Pearson Prentice Hall
3. F S Hillier, G J Liberman, Introduction to Operations Research, 9th edition, McGraw-Hill publication
4. Prem Kumar Gupta, Hira D S, Operations Research, 7th edition, S Chand Company PVT Ltd

OPERATIONS RESEARCH

UNIT 4

Lecture 3



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Lecture Details:

Basic Inventory Models – Price Discounts & Probabilistic Inventory Models

Operations Research/ME, 5 Semester.

LEARNING OUTCOMES

Upon completion of this lecture, student should be able to

- Determine optimal inventory policy for simple inventory problems with known demand

EOQ Model With Price Discounts

Procedure to Solve EOQ Problems with Price Discounts

- ❖ For each available unit cost c_j , use the EOQ formula for the EOQ model to calculate its optimal order quantity Q_j^*
- ❖ For each c_j where Q_j^* is within the feasible range of order quantities for c_j , calculate the corresponding total cost per unit time T_j
- ❖ For each c_j where Q_j^* is not within this feasible range, determine the order quantity Q_j that is at the endpoint of this feasible range that is closest to Q_j^* . Calculate the total cost per unit time T_j for Q_j and c_j
- ❖ Compare the T_j obtained for all the c_j and choose the minimum T_j . Then choose the order quantity Q_j obtained in step 2 or 3 that gives this minimum T_j .

EOQ Model With Price Discounts

Example 1

Annual demand for an item is 6000 units. Ordering cost is ₹600 per order. Inventory carrying cost is 18% of the purchase price/unit/year. The price break up is as shown below.

Quantity	Price
$0 \leq q_1 < 2000$	20
$2000 \leq q_2 < 4000$	15
$4000 \leq q_3$	9

EOQ Model With Price Discounts

$$Q_1^* = \sqrt{\frac{2DC_0}{C_h}} = \sqrt{\frac{2 \times 6000 \times 600}{0.18 \times 20}} = 1414.21$$

Since $0 < Q_1^* < 2000$,

$$T_1 = D \times c_1 + \frac{Q_1^*}{2} \times C_h + \frac{D}{Q_1^*} \times C_0$$

$$T_1 = 6000 \times 20 + \frac{1414.21}{2} \times 0.18 \times 20 + \frac{6000}{1414.21} \times 600$$

$$T_1 = ₹125091/\text{year}$$

EOQ Model With Price Discounts

$$Q_2^* = \sqrt{\frac{2DC_0}{C_h}} = \sqrt{\frac{2 \times 6000 \times 600}{0.18 \times 15}} = 1633$$

Since Q_2^* is not within the interval for c_2 , hence $Q_2 = 2000$

$$T_2 = D \times c_2 + \frac{Q_2}{2} \times C_h + \frac{D}{Q_2} \times C_0$$

$$T_2 = 6000 \times 15 + \frac{2000}{2} \times 0.18 \times 15 + \frac{6000}{2000} \times 600$$

$$T_2 = ₹94500/\text{year}$$

EOQ Model With Price Discounts

$$Q_3^* = \sqrt{\frac{2DC_0}{C_h}} = \sqrt{\frac{2 \times 6000 \times 600}{0.18 \times 9}} = 2108.2$$

Since Q_3^* is not within the interval for c_3 , hence $Q_3 = 4000$

$$T_3 = D \times c_3 + \frac{Q_3}{2} \times C_h + \frac{D}{Q_3} \times C_0$$

$$T_3 = 6000 \times 9 + \frac{4000}{2} \times 0.18 \times 9 + \frac{6000}{4000} \times 600$$

$$T_3 = ₹58140/\text{year}$$

EOQ Model With Price Discounts

Comparing total costs,

$$TC_{min} = T_3 = ₹58140/year$$

Optimum Order quantity = 4000 units

Number of orders

$$= \frac{6000}{4000} = 1.5 \text{ orders/year}$$

Optimum ordering cycle

$$= \frac{4000}{6000} = 0.67 \text{ years}$$

EOQ Model With Price Discounts

Example 2

The annual demand for a product is 500 units. The cost of storage per unit per year is 10% of the unit cost. The ordering cost is ₹180 for each order. The unit cost depends upon the amount ordered. The range of amount ordered and the unit cost price are as follows:

Quantity	Price
$0 \leq q_1 < 500$	25
$500 \leq q_2 < 1500$	24.80
$1500 \leq q_3 < 3000$	24.60
$3000 \leq q_4$	24.40

Determine the order quantity. (268 units)

Bibliography

1. S D Sharma, Operations Research – Theory, Methods and Applications, 15th edition, Kedar Nath Ram Nath
2. H A Taha, Operations Research – An Introduction, 8th edition, Pearson Prentice Hall
3. F S Hillier, G J Liberman, Introduction to Operations Research, 9th edition, McGraw-Hill publication
4. Prem Kumar Gupta, Hira D S, Operations Research, 7th edition, S Chand Company PVT Ltd

OPERATIONS RESEARCH

UNIT 4

Lecture 4



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Lecture Details:

Selective Inventory Control Techniques

Operations Research/ME, 5 Semester.

LEARNING OUTCOMES

Upon completion of this lecture, student should be able to

- Determine optimal inventory policy for simple inventory problems with known demand

Selective Inventory Control Techniques

ABC Analysis

- ☞ ABC analysis consists of separating the inventory items into three groups A, B and C.
 - ❖ A – High consumption value items
 - ❖ B – Moderate consumption value items
 - ❖ C – Low consumption value items

Selective Inventory Control Techniques

Category A Items

- ☞ These are the items which typically account for a large proportion of the overall value (investment), but a small percentage of the number of items.
- ☞ The items of this category are obtained frequently but in small number.
- ☞ A periodic review policy should be followed to minimise the shortage percentage of such items.
- ☞ Very strict inventory control is required for these items.

Selective Inventory Control Techniques

Category C Items

- ☞ These are the items which typically account for a large percentage of the number of items, but a small proportion of the overall value (investment).
- ☞ These items are usually stocked and very informal inventory control is observed.

Category B Items

- ☞ These items receive the intermediate treatment.

Selective Inventory Control Techniques

Example 1

The following information is known about items required for a company. Classify the items according to ABC Analysis.

Model Number	Annual consumption	Unit price in paisa
501	30000	10
502	280000	15
503	3000	10
504	110000	5
505	4000	5
506	220000	10

Model Number	Annual consumption	Unit price in paisa
507	15000	5
508	80000	5
509	60000	15
510	8000	10

Selective Inventory Control Techniques

Model Number	Annual consumption	Unit price in ₹	Usage Value In ₹	Ranking
501	30000	0.1	3000	6
502	280000	0.15	42000	1
503	3000	0.10	300	9
504	110000	0.05	5500	4
505	4000	0.05	200	10
506	220000	0.1	22000	2
507	15000	0.05	750	8
508	80000	0.05	4000	5
509	60000	0.15	9000	3
510	8000	0.10	800	7

Selective Inventory Control Techniques

Rank	Model	Number of units sold	Usage value	% of units sold	% of usage value
1	502	28000	42000	5.0	47.97
2	506	220000	22000	39.4	25.13
3	509	60000	9000	10.8	10.28
4	504	110000	5500	19.7	6.28
5	508	80000	4000	14.3	4.57
6	501	30000	3000	5.4	3.43
7	510	8000	800	1.4	0.91
8	507	15000	750	2.7	0.86
9	503	3000	300	0.5	0.34
10	505	4000	200	0.7	0.23
		558000	87550	5.0	47.97

Selective Inventory Control Techniques

Rank	Model	% of units sold	Cumulative % of units sold	% of usage value	Cumulative % of usage value	Category
1	502	5.0	5	47.97	48.0	A
2	506	39.4	44	25.13	73.1	
3	509	10.8	55	10.28	83.4	
4	504	19.7	75	6.28	89.7	B
5	508	14.3	89	4.57	94.2	
6	501	5.4	95	3.43	97.7	
7	510	1.4	96	0.91	98.6	
8	507	2.7	99	0.86	99.4	C
9	503	0.5	99	0.34	99.8	
10	505	0.7	100	0.23	100.0	

Selective Inventory Control Techniques

Technique	Basis of Classification	Uses
ABC	Consumption value	Control of raw material and WIP
VED	Criticality of items	Inventory levels of spare parts
XYZ	Values of items in storage	Review of inventory
FNSD	Consumption rate	Control of obsolescence

Bibliography

1. S D Sharma, Operations Research – Theory, Methods and Applications, 15th edition, Kedar Nath Ram Nath
2. H A Taha, Operations Research – An Introduction, 8th edition, Pearson Prentice Hall
3. F S Hillier, G J Liberman, Introduction to Operations Research, 9th edition, McGraw-Hill publication
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OPERATIONS RESEARCH

UNIT 4

Lecture 5



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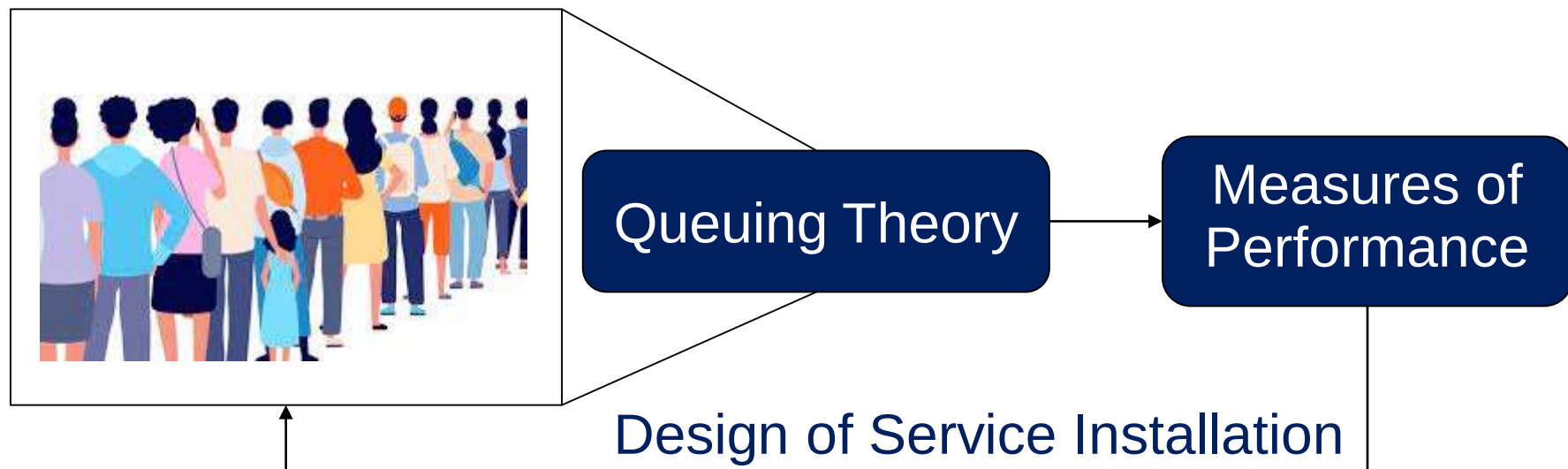
Lecture Details:

Queuing Systems - Introduction

Operations Research/ME, 5 Semester.

Introduction

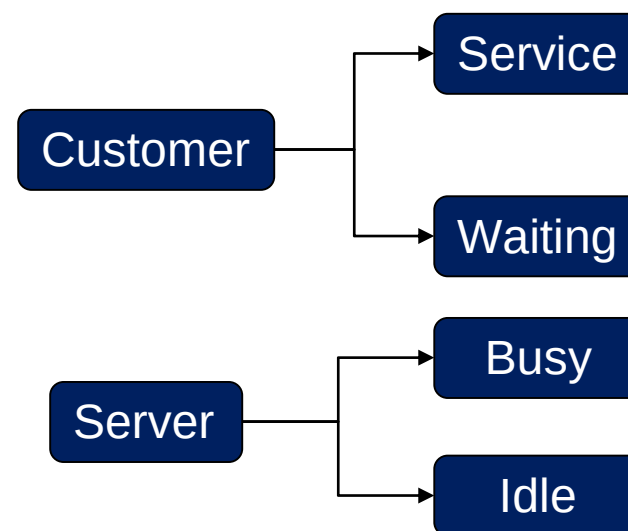
Queuing theory deals with the analysis of service facilities/waiting lines to offer a reasonably satisfactory service to waiting customers.



Elements of Queuing Model

The principal elements of a queuing system are the

CUSTOMER and the
SERVER



Elements of Queuing Model

Interarrival Time Used to describe arrival of customers

It is the time between the arrival of successive customers.

Service Time Used to describe service

It is the time the server takes to provide service to the customer.

Elements of Queuing Model

Queue Capacity

It refers to the number of customers allowed in the queue excluding the customer in service

Finite Queue Capacity



buffer area b/w machines

Infinite Queue Capacity



Elements of Queuing Model

Queue Discipline

It represents the order in which customers are selected from a queue.



First Come, First Served

FCFS



Last Come, First Served

LCFS



Service in Random Order

SIRO

Elements of Queuing Model

Queuing Behaviour

Jockeying

Moving from one queue to another in the hope of reducing waiting time

Balking

Not joining a queue altogether because of anticipated long delay

Reneging

Moving out from the queue because of long waiting time

Exponential Distribution

In queueing models, the random **inter-arrival** and **service times** are described quantitatively by **exponential distribution**.

$$f(t) = \lambda e^{-\lambda t}, t > 0$$

For an exponential distribution,

$$E\{t\} = 1/\lambda$$

$$P\{t \leq T\} = \int_0^T \lambda e^{-\lambda t} dt = 1 - e^{-\lambda T}$$

Exponential Distribution

Forgetfulness or Lack of Memory of the Exponential

If $f(t)$ represents the time t between the successive events, and S is the interval since the occurrence of the last event, then the forgetfulness property implies that

$$P\{t > T + S | t > S\} = P\{t > T\}$$

Exponential and Poisson Distribution

	Exponential	Poisson
Random variable	Time between successive arrivals, t	Number of arrivals, n , during a specified period T
Range	$t \geq 0$	$n = 0, 1, 2, \dots$
Density function	$f(t) = \lambda e^{-\lambda t}, t \geq 0$	$p_n(T) = \frac{(\lambda T)^n e^{-\lambda T}}{n!}, n = 0, 1, 2, \dots$
Mean value	$\frac{1}{\lambda}$ time units	λT arrivals during T
Cumulative probability	$P\{t \leq A\} = 1 - e^{-\lambda A}$	$p_{n \leq N}(T) = p_0(T) + p_1(T) + \dots + p_N(T)$
P{no arrivals during period A}	$p\{t > A\} = e^{-\lambda A}$	$p_0(A) = e^{-\lambda A}$

Generalized Poisson Queuing Model

- ❖ Both the inter-arrival and service times follow the exponential distribution.
- ❖ The queuing system is in a steady state i.e. the system is in operation for a sufficiently long time.
- ❖ Both the arrival and departure rates are state dependent, meaning that they depend on the number of customers in the service facility.

Generalized Poisson Queuing Model

Let

n = Number of customers in the system (queue + service)

λ_n = Arrival rate given n customers in the system

μ_n = Departure rate given n customers in the system

p_n = steady state probability of n customers in the system

Generalized Poisson Queuing Model

Steady state probability of n customers in the system

$$p_n = \left(\frac{\lambda_{n-1} \lambda_{n-2} \dots \lambda_0}{\mu_n \mu_{n-1} \dots \mu_1} \right) p_0, n = 1, 2, \dots$$

$$\sum_{n=0}^{\infty} p_n = 1$$

Queuing Models

The characteristics of the queuing system are given by the notation in the format

$(a/b/c):(d/e/f)$

a = Arrivals distribution

b = Departures (service time) distribution

c = Number of parallel servers (1,2,.... ∞)

d = Queue discipline

e = Maximum number allowed in the system

f = Size of the calling source

KENDALL NOTATION

$(d/e/f)$ → devised by A M Lee
 $(a/b/c)$ → devised by DG Kendall

Queuing Models

The standard notation for representing the arrivals and departure distributions (a and b) is

M = Markovian (or Poisson) arrivals or departure distribution

D = Constant (Deterministic) time

E_k = Erlang or Gamma distribution of time

GI = Generic distribution of inter-arrival time

G = Generic distribution of service time

Queuing Models

The queue discipline notation (d) includes

FCFS = First come, first served

LCFS = Last come, first served

SIRO = Service in random order

GD = General discipline (any type)

Example: $(M/M/10):(FCFS/20/\infty)$

Queuing model with Poisson arrival and service times, and 10 parallel servers.

The queue discipline is FCFS, and there is a limit of 20 customers on the system.

The size of the source from which customers arrive is infinite.

Steady State Measures of Performance

The most commonly used measures of performance in a queuing situation are

L_s = Expected number of customers in the system

L_q = Expected number of customers in the queue

W_s = Expected waiting time in the system

W_q = Expected waiting time in the queue

$\bar{c}$ = Expected number of busy servers

Steady State Measures of Performance

Let p_n be the steady state probability of n customers in the system, then

$$L_s = \sum_{n=1}^{\infty} np_n$$

$$L_q = \sum_{n=c+1}^{\infty} (n - c)p_n$$

Little's Formula

$$L_s = \lambda_{eff} W_s$$

$$L_q = \lambda_{eff} W_q$$

λ_{eff} is the **effective arrival rate**
at the system

Steady State Measures of Performance

$$\left(\begin{matrix} \text{Expected waiting} \\ \text{time in system} \end{matrix} \right) = \left(\begin{matrix} \text{Expected waiting} \\ \text{time in queue} \end{matrix} \right) + \left(\begin{matrix} \text{Expected service} \\ \text{time} \end{matrix} \right)$$

$\times \lambda_{eff}$
 From
 Little's formula

$$W_s = W_q + \frac{1}{\mu}$$

$$L_s = L_q + \frac{\lambda_{eff}}{\mu}$$

Average number of
busy servers

$$\bar{c} = L_s - L_q = \frac{\lambda_{eff}}{\mu}$$

Facility utilization

$$= \frac{\bar{c}}{c}$$

Single Server Model: (M/M/1):(GD/∞/ ∞)

Let

λ = Arrival rate of the customers (customers/unit time)

μ = Service rate (customers/unit time)

From the notation of the generalized model,

$$\left. \begin{array}{l} \lambda_n = \lambda \\ \mu_n = \mu \end{array} \right\} n = 0, 1, 2, \dots$$

Single Server Model: (M/M/1):(GD/∞/ ∞)

Since all arriving customers can join the system,

$$\lambda_{eff} = \lambda$$

Let

$$\rho = \frac{\lambda}{\mu}$$

Single Server Model: (M/M/1):(GD/∞/ ∞)

Expression for p_n is then equal to,

$$p_n = \left(\frac{\lambda_{n-1} \lambda_{n-2} \dots \lambda_0}{\mu_n \mu_{n-1} \dots \mu_1} \right) p_0, n = 0, 1, 2, \dots$$

$$p_n = \left(\frac{\lambda \times \lambda \times \dots \times \lambda}{\mu \times \mu \times \dots \times \mu} \right) p_0 = \left(\frac{\lambda}{\mu} \right)^n p_0 = \rho^n p_0$$

$$p_n = \rho^n p_0, n = 0, 1, 2, \dots$$

Single Server Model: (M/M/1):(GD/∞/ ∞)

$$\sum_{n=0}^{\infty} p_n = 1$$

$$p_0 + p_1 + p_2 + \dots = 1$$

$$p_0(1 + \rho + \rho^2 + \dots) = 1$$

$$p_0 = 1 - \rho \quad \text{if } \rho < 1$$

$$\therefore p_n = \rho^n(1 - \rho) \quad \text{if } \rho < 1$$

What if $\rho > 1$?

Single Server Model: (M/M/1):(GD/∞/ ∞)

$$\begin{aligned}L_s &= \sum_{n=0}^{\infty} np_n \\&= \sum_{n=0}^{\infty} n(1 - \rho)\rho^n \\&= (1 - \rho)\rho \frac{d}{d\rho} \left(\sum_{n=0}^{\infty} \rho^n \right) \\&= (1 - \rho)\rho \frac{d}{d\rho} \left(\frac{1}{1 - \rho} \right) \\ \therefore L_s &= \frac{\rho}{1 - \rho}\end{aligned}$$

Single Server Model: (M/M/1):(GD/∞/ ∞)

$$W_s = \frac{L_s}{\lambda} = \frac{1}{\mu(1 - \rho)} = \frac{1}{\mu - \lambda}$$

$$W_q = W_s - \frac{1}{\mu} = \frac{1}{\mu - \lambda} - \frac{1}{\mu} = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{\rho}{\mu(1 - \rho)}$$

$$L_q = \lambda W_q = \frac{\rho^2}{1 - \rho}$$

$$\bar{c} = L_s - L_q = \rho$$

Single Server Model: (M/M/1):(GD/N/ ∞)

In this model, there is a limit N on the number of customers allowed in the system i.e., maximum queue length = $N - 1$.

When the number of customers in the system reaches N , no more arrivals are allowed. Thus,

$$\lambda_n = \begin{cases} \lambda & n = 0, 1, \dots, N - 1 \\ 0 & n = N, N + 1, \dots \end{cases}$$

$$\mu_n = \mu, n = 0, 1, \dots$$

Single Server Model: (M/M/1):(GD/N/ ∞)

$$p_n = \begin{cases} \rho^n p_0, & n \leq N \\ 0, & n > N \end{cases}$$

$$\sum_{n=0}^{\infty} p_n = 1$$

$$p_0(1 + \rho + \rho^2 + \dots + \rho^N) = 1$$

Single Server Model: (M/M/1):(GD/N/ ∞)

$$\therefore p_0 = \begin{cases} \frac{1 - \rho}{1 - \rho^{N+1}}, & \rho \neq 1 \\ \frac{1}{N + 1} & \rho = 1 \end{cases}$$

Thus

$$p_n = \begin{cases} \frac{(1 - \rho)\rho^n}{1 - \rho^{N+1}}, & \rho \neq 1 \\ \frac{1}{N + 1}, & \rho = 1 \end{cases}, n = 0, 1, \dots, N$$

Single Server Model: (M/M/1):(GD/N/ ∞)

Because customers will be lost when there are N customers in the system,

$$\lambda_{lost} = \lambda p_N$$

$$\lambda_{eff} = \lambda - \lambda_{lost} = \lambda(1 - p_N)$$

Single Server Model: (M/M/1):(GD/N/ ∞)

The expected number of customers in the system is computed as

$$\begin{aligned} L_s &= \sum_{n=0}^N np_n \\ &= \left(\frac{1-\rho}{1-\rho^{N+1}} \right) \sum_{n=0}^N n\rho^n \\ &= \left(\frac{1-\rho}{1-\rho^{N+1}} \right) \rho \frac{d}{d\rho} \left(\sum_{n=0}^N \rho^n \right) \\ &= \left(\frac{1-\rho}{1-\rho^{N+1}} \right) \rho \frac{d}{d\rho} \left(\frac{1-\rho^{N+1}}{1-\rho} \right) \end{aligned}$$

Single Server Model: (M/M/1):(GD/N/ ∞)

The expected number of customers in the system is computed as

$$\begin{aligned}
 L_s &= \left(\frac{1 - \rho}{1 - \rho^{N+1}} \right) \rho \frac{d}{d\rho} \left(\frac{1 - \rho^{N+1}}{1 - \rho} \right) \\
 &= \left(\frac{1 - \rho}{1 - \rho^{N+1}} \right) \rho \left[\frac{(1 - \rho) \times -(N + 1)\rho^N - (1 - \rho^{N+1}) \times -1}{(1 - \rho)^2} \right] \\
 L_s &= \frac{\rho[1 - (N + 1)\rho^N + N\rho^{N+1}]}{(1 - \rho)(1 - \rho^{N+1})}, \rho \neq 1
 \end{aligned}$$

When $\rho = 1$, $L_s = N/2$

Bibliography

1. S D Sharma, Operations Research – Theory, Methods and Applications, 15th edition, Kedar Nath Ram Nath
2. H A Taha, Operations Research – An Introduction, 8th edition, Pearson Prentice Hall
3. F S Hillier, G J Liberman, Introduction to Operations Research, 9th edition, McGraw-Hill publication
4. Prem Kumar Gupta, Hira D S, Operations Research, 7th edition, S Chand Company PVT Ltd

OPERATIONS RESEARCH

UNIT 4

Lecture 6



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Lecture Details:

Queuing Systems - Introduction

Operations Research/ME, 5 Semester.

LEARNING OUTCOMES

Upon completion of this lecture, student should be able to

- Determine performance parameters of multi server queuing systems

Multiple Server Model: (M/M/c):(GD/∞/ ∞)

Let

λ = Arrival rate of the customers (customers/unit time)

μ = Service rate (customers/unit time)

From the notation of the generalized model,

$$\lambda_n = \lambda, \quad n \geq 0$$

$$\mu_n = \begin{cases} n\mu, & n < c \\ c\mu, & n \geq c \end{cases}$$

Multiple Server Model: (M/M/c):(GD/∞/ ∞)

$$p_n = \begin{cases} \frac{\rho^n}{n!} p_0, & n < c \\ \frac{\rho^n}{c! c^{n-c}} p_0, & n \geq c \end{cases}$$

$$p_0 = \left\{ \sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c}{c!} \left(\frac{1}{1 - \frac{\rho}{c}} \right) \right\}^{-1}, \frac{\rho}{c} < 1$$

$$L_q = \frac{\rho^{c+1}}{(c-1)! (c-\rho)^2} p_0$$

Multiple Server Model: (M/M/c):(GD/N/ ∞)

Let

λ = Arrival rate of the customers (customers/unit time)

μ = Service rate (customers/unit time)

From the notation of the generalized model,

$$\lambda_n = \begin{cases} \lambda, & 0 \leq n \leq N \\ 0, & n > N \end{cases}$$

$$\mu_n = \begin{cases} n\mu, & 0 \leq n < c \\ c\mu, & c \leq n \leq N \end{cases}$$

Multiple Server Model: (M/M/c):(GD/N/ ∞)

$$p_n = \begin{cases} \frac{\rho^n}{n!} p_0, & 0 \leq n < c \\ \frac{\rho^n}{c! c^{n-c}} p_0, & c \leq n \leq N \end{cases}$$

$$p_0 = \begin{cases} \left(\sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c \left(1 - \left(\frac{\rho}{c} \right)^{N-c+1} \right)}{c! \left(1 - \frac{\rho}{c} \right)} \right)^{-1}, & \frac{\rho}{c} \neq 1 \\ \left(\sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c}{c!} (N - c + 1) \right)^{-1}, & \frac{\rho}{c} = 1 \end{cases}$$

Multiple Server Model: (M/M/c):(GD/N/ ∞)

For $\frac{\rho}{c} \neq 1$

$$L_q = \frac{\rho^{c+1}}{(c-1)!(c-\rho)^2} \left\{ 1 - \left(\frac{\rho}{c}\right)^{N-c+1} - (N-c+1) \left(1 - \frac{\rho}{c}\right) \left(\frac{\rho}{c}\right)^{N-c} \right\} p_0$$

For $\frac{\rho}{c} = 1$

$$L_q = \frac{\rho^c (N-c)(N-c+1)}{2c!} p_0$$

$$\lambda_{lost} = \lambda p_N \text{ and } \lambda_{eff} = \lambda - \lambda_{lost}$$

Self Service Model: $(M/M/\infty):(GD/\infty/\infty)$

Let

λ = Arrival rate of the customers (customers/unit time)

μ = Service rate (customers/unit time)

From the notation of the generalized model,

$$\lambda_n = \lambda, n = 0, 1, 2, \dots$$

$$\mu_n = n\mu, n = 0, 1, 2, \dots$$

Self Service Model: (M/M/∞):(GD/∞/∞)

$$p_n = \frac{\lambda^n}{n! \mu^n} p_0 = \frac{\rho^n}{n!} p_0, n = 0, 1, 2, \dots$$

Where,

$$p_0 = e^{-\rho}$$

$$L_s = \sum_{n=0}^{\infty} n p_n = \sum_{n=0}^{\infty} \frac{\rho^n}{n-1!} p_0 = \rho$$

$$L_q = 0 \text{ and } W_q = 0$$

Bibliography

1. S D Sharma, Operations Research – Theory, Methods and Applications, 15th edition, Kedar Nath Ram Nath
2. H A Taha, Operations Research – An Introduction, 8th edition, Pearson Prentice Hall
3. F S Hillier, G J Liberman, Introduction to Operations Research, 9th edition, McGraw-Hill publication
4. Prem Kumar Gupta, Hira D S, Operations Research, 7th edition, S Chand Company PVT Ltd